

ADDITIONAL EXERCISES

EX 1:

a) $S_1 = \frac{1}{2} \quad S_2 = \frac{1}{2}$

THERE ARE 4 POSSIBILITIES

$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

$S: |S_1 - S_2|, |S_1 - S_2| + 1, \dots, S_1 + S_2 = 0, 1$

$M_S \leq |S|, -S \leq M_S \leq S$

$S = 1 \quad M_S = -1, 0, 1 \quad \text{TRIPLET}$

$S = 0 \quad M_S = 0 \quad \text{SINGLET}$

$S_z = S_z^1 + S_z^2 \rightarrow M_S = M_{S1} + M_{S2}$

$|\uparrow\uparrow\rangle \quad M_S = 1$

$|\uparrow\downarrow\rangle \quad M_S = 0$

$|\downarrow\uparrow\rangle \quad M_S = 0$

$|\downarrow\downarrow\rangle \quad M_S = -1$

APPLYING THE LOWERING OPERATOR $S_- = S_-^1 + S_-^2$ TO $|\uparrow\uparrow\rangle$:

$(S_-^1 + S_-^2) |\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$

THERE ARE THREE STATES WITH $S = 1$:

$$\begin{cases} |\uparrow\uparrow\rangle = |1, 1\rangle \\ \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = |1, 0\rangle \quad \text{TRIPLET} \\ |\downarrow\downarrow\rangle = |1, -1\rangle \end{cases}$$

AND ONE STATE WITH $S = 0$, FOR ORTHOGONALITY:

$\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) = |0, 0\rangle$

b) $\boxed{|1, 1\rangle} \quad J = 1 \quad M = M_1 + M_2 = 1 \rightarrow M_1 = M_2 = \frac{1}{2} \quad |\uparrow\uparrow\rangle$

FROM TABLE 2: $\langle J_1, \frac{1}{2}; m_1, m_2 | JM \rangle = \langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} | 1, 1 \rangle$

$C |\uparrow\uparrow\rangle$
 $C = \left(\frac{J_1 + m + \frac{1}{2}}{2J_1 + 1} \right)^{\frac{1}{2}} = \left(\frac{\frac{1}{2} + 1 + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = \left(\frac{2}{2} \right)^{\frac{1}{2}} = 1 \rightarrow 1 \cdot |\uparrow\uparrow\rangle$

$\boxed{|1, -1\rangle} \quad J = 1 \quad M = M_1 + M_2 = -1 \rightarrow M_1 = M_2 = -\frac{1}{2} \quad |\downarrow\downarrow\rangle$

$C |\downarrow\downarrow\rangle \quad C = \left(\frac{\frac{1}{2} - (-1) + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = 1 \rightarrow 1 \cdot |\downarrow\downarrow\rangle$

$\boxed{|1, 0\rangle} \quad J = 1 \quad M = 0 \quad \begin{cases} m_1 = \frac{1}{2} & m_2 = -\frac{1}{2} & |\uparrow\downarrow\rangle \\ m_2 = -\frac{1}{2} & m_1 = \frac{1}{2} & |\downarrow\uparrow\rangle \end{cases}$

$c_1 |\uparrow\downarrow\rangle + c_2 |\downarrow\uparrow\rangle$
 $c_1 = \left(\frac{\frac{1}{2} - 0 + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = \left(\frac{1}{2} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$
 $c_2 = \left(\frac{\frac{1}{2} + 0 + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$\boxed{|0, 0\rangle} \quad J = 0 \quad M = 0 \quad \begin{cases} m_1 = \frac{1}{2} & m_2 = -\frac{1}{2} & |\uparrow\downarrow\rangle \\ m_1 = -\frac{1}{2} & m_2 = \frac{1}{2} & |\downarrow\uparrow\rangle \end{cases}$

$c_1 |\uparrow\downarrow\rangle + c_2 |\downarrow\uparrow\rangle$
 $c_1 = \left(\frac{\frac{1}{2} + 0 + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$
 $c_2 = - \left(\frac{\frac{1}{2} - 0 + \frac{1}{2}}{2 \cdot \frac{1}{2} + 1} \right)^{\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad \Rightarrow \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

these are the same coefficients found in part a)

$$c) |l_1, l_2, l, m\rangle = |1, 1, 0, 0\rangle$$

(1)

$$J_1 = 1 \quad m_1 = -1, 0, 1$$

$$J_2 = 1 \quad m_2 = -1, 0, 1$$

$$|J_1 - J_2| = 0 \quad J_1 + J_2 = 1 \Rightarrow J = 0, 1$$

$$J = 0 \quad m = 0, \quad m = m_1 + m_2$$

$$\begin{cases} m_1 = 1 & m_2 = -1 & |1, -1\rangle \\ m_2 = -1 & m_2 = 1 & |-1, 1\rangle \\ m_1 = 0 & m_2 = 0 & |0, 0\rangle \end{cases}$$

from table 2:

$$J = 0 \quad J_2 = 1$$

$$c_1 |1, -1\rangle + c_2 |-1, 1\rangle + c_3 |0, 0\rangle$$

$$C_1: \langle 1, 1; 1, -1 | 0, 0 \rangle$$

$$C_1 = \left[\frac{(1+0)(1+0+1)}{2 \cdot 1 (2 \cdot 1 + 1)} \right]^{\frac{1}{2}} = \left(\frac{2}{6} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

$$C_2: \langle 1, 1; -1, 1 | 0, 0 \rangle$$

$$C_2 = \left[\frac{(1-0)(1-0+1)}{2 \cdot 1 (2 \cdot 1 + 1)} \right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

$$C_3: \langle 1, 1; 0, 0 | 0, 0 \rangle$$

$$C_3 = - \left[\frac{(1-0)(1+0)}{1 (2+1)} \right] = - \frac{1}{\sqrt{3}}$$

EX 2 (a)

$$\Psi = \Psi_{\text{SPATIAL}} \cdot \chi_{\text{SPIN}}$$

Since electrons are fermions, their wavefunction Ψ must be antisymmetric

- for triplet states, χ_{SPIN} is symmetric and Ψ_{SPATIAL} must be antisymmetric:

$$(-1)^{l-l_1-l_2} = -1$$

$l_1 = l_2$ (the electrons are in equivalent orbitals)

$$(-1)^{l-2l_1} = -1 \rightarrow l-2l_1 = \text{ODD} \rightarrow \begin{matrix} l = \text{ODD} \\ (2l_1 \text{ IS EVEN}) \end{matrix}$$

- for the singlet state χ_{SPIN} is antisymmetric and Ψ_{SPATIAL} must be symmetric

$$(-1)^{l-2l_1} = 1 \rightarrow l-2l_1 = \text{EVEN} \rightarrow l = \text{EVEN}$$

b) $\langle l_1, l_2; m_1-1, m_2+1 | l, l \rangle$

$$m_1 \rightarrow m_1-1 \quad L_1^- \text{ OPERATOR}$$

$$m_2 \rightarrow m_2+1 \quad L_2^+ \text{ OPERATOR}$$

$$L_1^- \langle l_1, m_1 | l_1, l \rangle = \hbar \sqrt{l_1(l_1+1) - m_1(m_1-1)} \langle l_1, m_1-1 | l_1, l \rangle$$

$$\rightarrow \langle l_1, m_1-1 | l_1, l \rangle = \frac{L_1^- \langle l_1, m_1 | l_1, l \rangle}{\hbar \sqrt{l_1(l_1+1) - m_1(m_1-1)}}$$

$$L_2^+ \langle l_2, m_2 | l_2, l \rangle = \hbar \sqrt{l_2(l_2+1) - m_2(m_2+1)} \langle l_2, m_2+1 | l_2, l \rangle$$

$$\frac{1}{\hbar \sqrt{l_1(l_1+1) - m_1(m_1-1)}} \cdot L_1^- \langle l_1, m_1 | l_1, l \rangle \times L_2^+ \langle l_2, m_2 | l_2, l \rangle$$

$$= \frac{\sqrt{l_2(l_2+1) - m_2(m_2+1)}}{\sqrt{l_1(l_1+1) - m_1(m_1-1)}} L_1^- L_2^+ \langle l_1, l_2; m_1, m_2 | l, l \rangle$$

$$\frac{\sqrt{l_2(l_2+1) - m_2(m_2+1)}}{\sqrt{l_1(l_1+1) - m_1(m_1-1)}} \langle l_1, l_2; m_1, m_2 | l, l \rangle = - \langle l_1, l_2; m_1-1, m_2+1 | l, l \rangle$$

and

$$\langle l_1, l_2; m_1-1, m_2+1 | \ell \ell \rangle = - \sqrt{\frac{l_2(l_2+1) - m_2(m_2+1)}{l_1(l_1+1) - m_1(m_1-1)}} \langle l_1, l_2; m_1, m_2 | \ell \ell \rangle$$

c) verify for $\langle l_1, l_2; m_1, m_2 | \ell \ell \rangle = \langle 1, \frac{1}{2}; 1, -\frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle$
the relation becomes:

$$\langle 1, \frac{1}{2}; 0, \frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle = - \sqrt{\frac{\frac{1}{2}(\frac{3}{2}) + \frac{1}{2}(\frac{1}{2})}{1 \cdot 2 - 1(0)}} \langle 1, \frac{1}{2}; 1, -\frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle$$

$$\langle 1, \frac{1}{2}; 0, \frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle = -\frac{1}{\sqrt{3}}$$

($1 \times \frac{1}{2}$ TABLE $m_1=0$ $m_2=\frac{1}{2}$)

$$\langle 1, \frac{1}{2}; 1, -\frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle = \sqrt{\frac{2}{3}}$$

($m_1=1$ $m_2=-\frac{1}{2}$)

$$-\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \rightarrow \text{the relation holds}$$

Ex 3:

$$a) \cdot S = \frac{1}{2}$$

$$m_s = -\frac{1}{2}; +\frac{1}{2}$$

TWO POSSIBLE STATES: $|\frac{1}{2} \frac{1}{2}\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\frac{1}{2} -\frac{1}{2}\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S^2 |\uparrow\rangle = \hbar^2 S(S+1) |\uparrow\rangle = \frac{3}{4} \hbar^2 |\uparrow\rangle \quad \text{SINCE } S = \frac{1}{2}$$

IN MATRIX NOTATION:

$$a = \frac{3}{4} \hbar^2$$

$$c = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{4} \hbar^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \frac{3}{4} \hbar^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$$

$$S^2 |\downarrow\rangle = \frac{3}{4} \hbar^2 |\downarrow\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{4} \hbar^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \frac{3}{4} \hbar^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} b = 0 \\ d = \frac{3}{4} \hbar^2 \end{matrix}$$

$$S^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z |\uparrow\rangle = m_s \hbar |\uparrow\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +\frac{1}{2} \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = +\frac{1}{2} \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c = 0 \\ a = +\frac{1}{2} \hbar \end{matrix}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_z |\downarrow\rangle = m_s \hbar |\downarrow\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2} \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} b \\ d \end{bmatrix} = -\frac{1}{2} \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} b = 0 \\ d = -\frac{1}{2} \hbar \end{matrix}$$

$$\left. \begin{matrix} S_+ = S_x + i S_y \\ S_- = S_x - i S_y \end{matrix} \right\} \Rightarrow$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

$$S_+ |\downarrow\rangle = \hbar |\uparrow\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b = \hbar \\ d = 0 \end{matrix}$$

$$S_+ |\uparrow\rangle = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ c = 0 \end{matrix}$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- |\uparrow\rangle = \hbar |\downarrow\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ c = \hbar \end{matrix}$$

$$S_- |\downarrow\rangle = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b = 0 \\ d = 0 \end{matrix}$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S = \frac{3}{2} \quad -\frac{3}{2} \leq m_s \leq \frac{3}{2} \quad m_s = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$$

$$S^2 \chi = \hbar^2 S(S+1) \chi \Rightarrow S^2 \chi = \frac{15}{4} \hbar^2 \chi$$

4 SPIN STATES : S^2 IS A 4×4 MATRIX

$$S^2 \chi_1 = \frac{15}{4} \hbar^2 \chi_1$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\chi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & l & m & n \\ o & p & q & r \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{15}{4} \hbar^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ e \\ i \\ o \end{bmatrix} = \frac{15}{4} \hbar^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow a = \frac{15}{4} \hbar^2, e = i = o = 0$$

SIMILARLY:

$$\begin{bmatrix} b \\ f \\ e \\ p \end{bmatrix} = \frac{15}{4} \hbar^2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow f = \frac{15}{4} \hbar^2, b = e = p = 0$$

$$\begin{bmatrix} c \\ g \\ m \\ q \end{bmatrix} = \frac{15}{4} \hbar^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow m = \frac{15}{4} \hbar^2, c = g = q = 0$$

$$\begin{bmatrix} d \\ h \\ n \\ r \end{bmatrix} = \frac{15}{4} \hbar^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow r = \frac{15}{4} \hbar^2, d = h = n = 0$$

$$S^2 = \frac{15}{4} \hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_z \chi = m_s \hbar \chi$$

$$S_z \chi_1 = \frac{3}{2} \hbar \chi_1$$

$$S_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{2} \hbar \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ e \\ i \\ o \end{bmatrix} = \frac{3}{2} \hbar \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow a = \frac{3}{2} \hbar, e = i = o = 0$$

$$S_z \chi_2 = \frac{1}{2} \hbar \chi_2$$

$$S_z \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} b \\ f \\ e \\ p \end{bmatrix} = \frac{1}{2} \hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow f = \frac{1}{2} \hbar, b = e = p = 0$$

$$S_z \chi_3 = -\frac{1}{2} \hbar \chi_3$$

$$S_z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c \\ g \\ m \\ q \end{bmatrix} = -\frac{1}{2} \hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow m = -\frac{1}{2} \hbar, c = g = q = 0$$

$$S_z = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & l & m & n \\ o & p & q & r \end{bmatrix}$$

$$S_z \chi_1 = -\frac{3}{2} \hbar \chi_1$$

$$S_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{3}{2} \hbar \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} d \\ b \\ n \\ r \end{bmatrix} = -\frac{3}{2} \hbar \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow r = -\frac{3}{2} \hbar, d = b = n = 0$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$L_{\pm} |l m\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle$$

$$S_+ \chi_1 = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - \frac{3}{2}(\frac{5}{2})} = 0 \quad \chi_1 = |\frac{3}{2} \frac{3}{2}\rangle$$

$$S_+ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ e \\ i \\ o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad a = e = i = o = 0$$

$$S_+ = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & l & m & n \\ o & p & q & r \end{bmatrix}$$

$$S_+ \chi_2 = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - \frac{1}{2}(\frac{3}{2})} \chi_1 \quad \chi_2 = |\frac{3}{2} \frac{1}{2}\rangle$$

$$S_+ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \hbar \sqrt{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} b \\ f \\ l \\ e \end{bmatrix} = \hbar \sqrt{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow b = \hbar \sqrt{3}, f = l = e = 0$$

$$S_+ \chi_3 = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - [-\frac{1}{2}(\frac{1}{2})]} \chi_2 \quad \chi_3 = |\frac{3}{2} -\frac{1}{2}\rangle$$

$$S_+ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \hbar 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c \\ g \\ m \\ q \end{bmatrix} = 2 \hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow g = 2 \hbar, c = m = q = 0$$

$$S_+ \chi_4 = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - [-\frac{3}{2}(-\frac{1}{2})]} \chi_3$$

$$S_+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \hbar \sqrt{3} \hbar \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} d \\ h \\ n \\ r \end{bmatrix} = \sqrt{3} \hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow n = \sqrt{3} \hbar, d = h = r = 0$$

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_- \chi_1 = \hbar \sqrt{\frac{3}{2} \left(\frac{5}{2}\right) - \frac{3}{2} \left(\frac{1}{2}\right)} \chi_2$$

$$S_- \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{3} \hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ e \\ 1 \\ 0 \end{bmatrix} = \sqrt{3} \hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} e = \sqrt{3} \hbar \\ a = i = 0 = 0 \end{matrix}$$

$$S_- \chi_2 = \hbar \sqrt{\frac{3}{2} \left(\frac{5}{2}\right) - \frac{1}{2} \left(-\frac{3}{2}\right)} \chi_3$$

$$S_- \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2 \hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} b \\ f \\ 1 \\ p \end{bmatrix} = 2 \hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} e = 2 \hbar \\ b = f = p = 0 \end{matrix}$$

$$S_- \chi_3 = \hbar \sqrt{\frac{3}{2} \left(\frac{5}{2}\right) + \frac{1}{2} \left(-\frac{3}{2}\right)} \chi_4$$

$$S_- \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{3} \hbar \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} c \\ g \\ 0 \\ q \end{bmatrix} = \sqrt{3} \hbar \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} q = \sqrt{3} \hbar \\ c = g = u = 0 \end{matrix}$$

$$S_- \chi_4 = 0$$

$$S_- \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} d \\ h \\ n \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow d = h = n = r = 0$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2} = \frac{1}{2} \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y = \frac{S_+ - S_-}{2i} = \frac{1}{2i} \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} = \frac{1}{2} \hbar \begin{pmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}$$

$$b) [S_x, S_y] = S_x S_y - S_y S_x$$

$$[S_x, S_y] = \frac{1}{4} \hbar^2 \begin{pmatrix} 3i & 0 & -2\sqrt{3}i & 0 \\ 0 & i & 0 & -2\sqrt{3}i \\ 2\sqrt{3}i & 0 & -i & 0 \\ 0 & 2\sqrt{3}i & 0 & -3i \end{pmatrix} - \frac{1}{4} \hbar^2 \begin{pmatrix} -3i & 0 & -2\sqrt{3}i & 0 \\ 0 & -i & 0 & -2\sqrt{3}i \\ 2\sqrt{3}i & 0 & i & 0 \\ 0 & 2\sqrt{3}i & 0 & 3i \end{pmatrix}$$

$$[S_x, S_y] = \frac{1}{4} \hbar^2 \begin{pmatrix} 6i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & -6i \end{pmatrix} = i \hbar^2 \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = i \hbar^2 S_z$$

$$c) S = \frac{1}{2}$$

$$\langle S \cdot \hat{n} \rangle = \pm \frac{\hbar}{2} \quad \hat{n} \text{ ARBITRARY DIRECTION}$$

$$\hat{n} = \hat{x} = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k} \quad \text{SPHERICAL COORD.}$$

$$S \cdot \hat{n} = S_x \sin\theta \cos\phi \hat{i} + S_y \sin\theta \sin\phi \hat{j} + S_z \cos\theta \hat{k}$$

TAKING THE RESULTS IN PART a)

$$S \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta \cos\phi + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\theta \sin\phi + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta$$

$$S \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} 0 & \sin\theta \cos\phi \\ \sin\theta \cos\phi & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \sin\theta \sin\phi \\ i \sin\theta \sin\phi & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

$$S \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta (\cos\phi - i \sin\phi) \\ \sin\theta (\cos\phi + i \sin\phi) & -\cos\theta \end{pmatrix} \quad e^{i\alpha} = \cos\alpha + i \sin\alpha$$

$$S \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & e^{-i\phi} \sin\theta \\ e^{i\phi} \sin\theta & -\cos\theta \end{pmatrix}$$

$$\langle S \cdot \hat{n} \rangle = \langle \chi^\dagger | S \cdot \hat{n} | \chi \rangle$$

EIGENVALUES AND EIGENSPINOR χ OF $\langle S \cdot \hat{n} \rangle$:

EIGENVALUES:

$$\begin{vmatrix} \frac{\hbar}{2} \cos\theta - \lambda & \frac{\hbar}{2} e^{-i\phi} \sin\theta \\ \frac{\hbar}{2} e^{i\phi} \sin\theta & -\frac{\hbar}{2} \cos\theta - \lambda \end{vmatrix} = 0 \quad -\frac{\hbar^2}{4} \cos^2\theta + \lambda^2 - \frac{\hbar^2}{4} \sin^2\theta = 0$$

$$\frac{\hbar^2}{4} (\sin^2\theta + \cos^2\theta) - \lambda^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

$$\text{EIGENSPINORS: } S \cdot \hat{n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} \cos\theta & e^{-i\phi} \sin\theta \\ e^{i\phi} \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} \alpha \cos\theta & +\beta e^{i\phi} \sin\theta \\ \alpha e^{i\phi} \sin\theta & -\beta \cos\theta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\pm \alpha = \alpha \cos \theta + \beta e^{-i\phi} \sin \theta$$

$$\pm \beta = \alpha e^{i\phi} \sin \theta - \beta \cos \theta \rightarrow \beta (\pm 1 + \cos \theta) = \alpha e^{i\phi} \sin \theta$$

$$\beta e^{i\phi} \sin \theta = \pm \alpha (1 \mp \cos \theta)$$

$$\beta = \frac{\alpha e^{i\phi} \sin \theta}{\pm 1 + \cos \theta}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$$\text{④ } \beta e^{-i\phi} \sin \theta = \alpha (1 - \cos \theta)$$

$$\beta = \frac{\alpha e^{i\phi} \sin \theta}{1 + \cos \theta} = \alpha e^{i\phi} \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \alpha e^{i\phi} \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$$

$$\text{NORMATIZATION } |\alpha|^2 + |\beta|^2 = 1$$

$$|\alpha|^2 + |\alpha|^2 \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} = 1$$

$$|\alpha|^2 \left(1 + \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} \right) = 1$$

$$|\alpha|^2 \cdot \frac{1}{\cos^2 \frac{\theta}{2}} = 1 \rightarrow \alpha = \cos \frac{\theta}{2}, \beta = e^{i\phi} \sin \frac{\theta}{2}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$\chi^+ = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}$$

$$\text{⑤ } \beta = \frac{\alpha e^{i\phi} \sin \theta}{\cos \theta - 1} = \alpha e^{i\phi} \cdot \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{-2 \sin^2 \frac{\theta}{2}} = -\alpha e^{i\phi} \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$|\alpha|^2 + |\alpha|^2 \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} = 1$$

$$|\alpha|^2 \left(1 + \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \right) = 1 \quad |\alpha|^2 \frac{1}{\sin^2 \frac{\theta}{2}} = 1$$

$$\alpha = \sin \frac{\theta}{2}, \beta = -e^{i\phi} \cos \frac{\theta}{2}, \chi^- = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}$$

$$\langle S \cdot \hat{n} \rangle = \langle \chi^+ | S \cdot \hat{n} | \chi \rangle$$

$$\text{⑥ } \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} & e^{-i\phi} \sin \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix} =$$

$$= \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} \cos \theta + \sin \theta \sin \frac{\theta}{2} & e^{-i\phi} \sin \theta \cos \frac{\theta}{2} - e^{-i\phi} \cos \theta \sin \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix}$$

$$= \frac{\hbar}{2} \left[\cos^2 \frac{\theta}{2} \cos \theta + \sin \theta \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sin \theta \sin \frac{\theta}{2} \cos \frac{\theta}{2} - \cos \theta \sin^2 \frac{\theta}{2} \right]$$

$$= \frac{\hbar}{2} \left[\underbrace{\sin \theta \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}_{\sin \theta} + \cos \theta \underbrace{\left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right)}_{1 - 2 \sin^2 \frac{\theta}{2} = \cos \theta} \right] = \frac{\hbar}{2} (\sin^2 \theta + \cos^2 \theta) = \frac{\hbar}{2}$$

⑦ in the same way we find

$$\langle \chi^- | S \cdot \hat{n} | \chi \rangle = -\frac{\hbar}{2}$$

Ex 4

a) $H = H_0 + \lambda H'$

$H_0 | \phi_n^{(0)} \rangle = E_n^{(0)} | \phi_n^{(0)} \rangle$

$E_n^{(0)}$ = unperturbed energy levels

power series of ϕ_n and E_n

$\phi_n = \phi_n^{(0)} + \lambda \phi_n^{(1)} + \lambda^2 \phi_n^{(2)} + \dots$

$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$

$H \phi_n = E_n \phi_n \rightarrow (H_0 + \lambda H') (\phi_n^{(0)} + \lambda \phi_n^{(1)} + \lambda^2 \phi_n^{(2)} + \dots) = (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots) (\phi_n^{(0)} + \lambda \phi_n^{(1)} + \lambda^2 \phi_n^{(2)} + \dots)$

$\Rightarrow H_0 \phi_n^{(0)} + \lambda (H_0 \phi_n^{(1)} + H' \phi_n^{(0)}) + \lambda^2 (H_0 \phi_n^{(2)} + H' \phi_n^{(1)}) + \dots = E_n^{(0)} \phi_n^{(0)} + \lambda (E_n^{(0)} \phi_n^{(1)} + E_n^{(1)} \phi_n^{(0)}) + \lambda^2 (E_n^{(0)} \phi_n^{(2)} + E_n^{(1)} \phi_n^{(1)} + E_n^{(2)} \phi_n^{(0)}) + \dots$

FIRST ORDER CORRECTION

$H_0 \phi_n^{(1)} + H' \phi_n^{(0)} = E_n^{(0)} \phi_n^{(1)} + E_n^{(1)} \phi_n^{(0)}$ (coeff. of λ)

inner product $\langle \phi_n^{(0)} | H | \phi_n^{(0)} \rangle$

$\langle \phi_n^{(0)} | H_0 | \phi_n^{(1)} \rangle + \langle \phi_n^{(0)} | H' | \phi_n^{(0)} \rangle = E_n^{(0)} \langle \phi_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(1)} \underbrace{\langle \phi_n^{(0)} | \phi_n^{(0)} \rangle}_1$

H_0 is Hermitian:

$\langle \phi_n^{(0)} | H_0 | \phi_n^{(1)} \rangle = \langle H_0 \phi_n^{(0)} | \phi_n^{(1)} \rangle = E_n^{(0)} \langle \phi_n^{(0)} | \phi_n^{(1)} \rangle$

$\Rightarrow \boxed{\langle \phi_n^{(0)} | H' | \phi_n^{(0)} \rangle = E_n^{(1)}}$

SECOND ORDER CORRECTION (coeff. of λ^2)

$H_0 \phi_n^{(2)} + H' \phi_n^{(1)} = E_n^{(0)} \phi_n^{(2)} + E_n^{(1)} \phi_n^{(1)} + E_n^{(2)} \phi_n^{(0)}$

inner product:

$\langle \phi_n^{(0)} | H_0 | \phi_n^{(2)} \rangle + \langle \phi_n^{(0)} | H' | \phi_n^{(1)} \rangle = E_n^{(0)} \langle \phi_n^{(0)} | \phi_n^{(2)} \rangle + E_n^{(1)} \langle \phi_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(2)}$

H_0 is Hermitian:

$\langle \phi_n^{(0)} | H_0 | \phi_n^{(2)} \rangle = \langle H_0 \phi_n^{(0)} | \phi_n^{(2)} \rangle = E_n^{(0)} \langle \phi_n^{(0)} | \phi_n^{(2)} \rangle$

$\Rightarrow \langle \phi_n^{(0)} | H' | \phi_n^{(1)} \rangle = E_n^{(1)} \langle \phi_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(2)}$

$\langle \phi_n^{(0)} | H' | \phi_n^{(1)} \rangle = E_n^{(2)} = 0$ since $\langle \phi^{(m)} | \phi^{(n)} \rangle = \delta_{mn}$ (0 if $m \neq n$, 1 if $m = n$)

$E_n^{(2)} = \sum_{m \neq n} c_m^{(n)} \langle \phi_n^{(0)} | H' | \phi_m^{(0)} \rangle = \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle \langle \psi_n | H' | \psi_m \rangle}{E_n^{(0)} - E_m^{(0)}}$

$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \sum_{m \neq n} \frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$

b) FINE STRUCTURE

$$E_n^{(1)} = \langle \Phi_n^{(0)} | H' | \Phi_n^{(0)} \rangle$$

$$H' = H'_1 + H'_2 + H'_3$$

$$\langle \Phi_n^{(0)} | H'_1 | \Phi_n^{(0)} \rangle$$

$$E_n^{(1)} = \langle \Phi_n^{(0)} | H'_1 | \Phi_n^{(0)} \rangle + \langle \Phi_n^{(0)} | H'_2 | \Phi_n^{(0)} \rangle + \langle \Phi_n^{(0)} | H'_3 | \Phi_n^{(0)} \rangle$$

$$H'_1 = \frac{-p^4}{8m^3c^2}$$

$$\langle \Phi_n^{(0)} | H'_1 | \Phi_n^{(0)} \rangle = -\frac{1}{8m^3c^2} \langle \Phi_n^{(0)} | p^4 | \Phi_n^{(0)} \rangle = -\frac{1}{8m^3c^2} \langle p^2 \Phi_n^{(0)} | p^2 \Phi_n^{(0)} \rangle$$

$$p^2 \Phi_n^{(0)} = 2m(E - V) \Phi_n^{(0)}$$

$$\langle \Phi_n^{(0)} | H'_1 | \Phi_n^{(0)} \rangle = -\frac{1}{8m^3c^2} \cdot 4m^2 \langle (E_n^{(0)} - V)^2 \rangle = -\frac{1}{2mc^2} [E_n^{(0)} - 2E_n^{(0)} \langle V \rangle + \langle V^2 \rangle]$$

$$= -\frac{1}{2mc^2} \left[E_n^{(0)2} + \frac{2E_n^{(0)} \pm e^2}{4\pi\epsilon_0} \left\langle \frac{1}{r} \right\rangle + \left(\frac{Ze^2}{4\pi\epsilon_0} \right)^2 \left\langle \frac{1}{r^2} \right\rangle \right]$$

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$$

$$= -\frac{1}{2mc^2} \left[E_n^{(0)2} + \frac{2E_n^{(0)} \pm e^2}{4\pi\epsilon_0} \frac{Z}{n^2 a_0} + \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{Z^2}{n^3 a_0^2 (l+1/2)} \right] \quad \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$$

$$= -\frac{1}{2mc^2} \left[(E_n^{(0)})^2 - 4E_n^{(0)} \cdot E_n^{(0)} + \frac{(E_n^{(0)})^2}{na_0(l+1/2)} \right]$$

$$= -\frac{1}{2mc^2} \left[-3(E_n^{(0)})^2 + \frac{E_n^{(0)2}}{na_0(l+1/2)} \right] = \boxed{-\frac{(E_n^{(0)})^2}{2mc^2} \left[\frac{4n}{l+1/2} - 3 \right] = \langle H'_1 \rangle}$$

$$H'_2 = \frac{1}{2m^2c^2 r} \frac{dV}{dr} L \cdot S$$

$$l+1/2 = j$$

$$\frac{dV}{dr} = \frac{Ze^2}{4\pi\epsilon_0 r^2} \Rightarrow \langle H'_2 \rangle = \frac{Ze^2}{8\pi\epsilon_0 m^2 c^2} \left\langle \frac{1}{r^3} L \cdot S \right\rangle$$

j is conserved

$$J = L + S \quad J^2 = (L+S) \cdot (L+S) = L^2 + S^2 + 2LS$$

$$L \cdot S = \frac{1}{2} (J^2 - L^2 - S^2)$$

eigenvalues of $L \cdot S$: $\frac{\hbar^2}{2} [j(j+1) - l(l+1) - s(s+1)]$ with $s = \frac{1}{2}$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{l^3}{n^3 a_0^3 l(l+1/2)(l+1)}$$

$$\Rightarrow \langle H'_2 \rangle = \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \cdot \frac{\hbar^2}{2} \frac{j(j+1) - l(l+1) - \frac{3}{4}}{l(l+1/2)(l+1)} =$$

$$= \frac{(E_n^{(0)})^2}{mc^2} \left(\frac{n[j(j+1) - l(l+1) - \frac{3}{4}]}{l(l+1/2)(l+1)} \right) \quad \begin{matrix} j = l + \frac{1}{2} \\ l = j - \frac{1}{2} \end{matrix}$$

$$= \boxed{\frac{(E_n^{(0)})^2}{2mc^2} \left(3 - \frac{4n}{j+1/2} \right) = \langle H'_2 \rangle}$$

$$H'_3 = \frac{\pi \hbar^2}{2m^2 c^2} \frac{\pm e^2}{4\pi\epsilon_0} \sigma_z$$

$$\langle H'_3 \rangle = \frac{\pi \hbar^2}{2m^2 c^2} \frac{\pm e^2}{4\pi\epsilon_0} \langle \sigma_z \rangle = \frac{\pi \hbar^2}{2m^2 c^2} \frac{\pm e^2}{4\pi\epsilon_0} |\Psi_{n00}^{(0)}(r=\infty)|^2$$

$$= \frac{\pi \hbar^2}{2m^2 c^2} \frac{\pm e^2}{4\pi\epsilon_0} \frac{Z^3}{\pi a_0^3 n^3} = \boxed{-\frac{\hbar^2}{m^2 c^2} \frac{Z^2}{a_0^2 n} E_n^{(0)} = \langle H'_3 \rangle}$$

$$= -\frac{\hbar^2}{m^2 c^2} \frac{Z^2 m e^2}{a_0 n 4\pi\epsilon_0 \hbar^2} E_n^{(0)}$$

$$\langle H' \rangle = \langle H_1' \rangle + \langle H_2' \rangle + \langle H_3' \rangle$$

$$\langle H' \rangle = \frac{(E_n^{(0)})^2}{2mc^2} \left[3 - \frac{4n}{j} + 3 - \frac{4n}{j+\frac{1}{2}} + 4n \right] =$$

$$= E_n^{(0)} \cdot \frac{E_n^{(0)}}{2mc^2} \left[6 - \frac{4n}{j} - \frac{4n}{j+\frac{1}{2}} + 4n \right] = E_n^{(0)} \frac{Z^4 e^4}{64 \pi^2 \epsilon_0^2 \hbar^2 c^2 n^2} \left(\frac{4n}{j+\frac{1}{2}} - 3 \right)$$

$$\langle H' \rangle = E_n^{(1)} = E_n^{(0)} \frac{(Z\alpha)^2}{n^2} \left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right)$$

The fine structure is smaller by a factor α^2 with respect to the Bohr energies:

order of magnitude :

$$\frac{\langle H' \rangle}{mc^2} \approx \alpha^4$$

$E \propto S$

$$a) E_{nf}^{(1)} = \langle \psi_{nf} | \hat{H}' | \psi_{nf} \rangle = -\frac{2}{3\epsilon_0 c^2} \langle \mu_e \cdot \mu_p \delta^3(r) \rangle$$

$$\langle \mu_e \cdot \mu_p \rangle = -\frac{g_s g_p e^2}{4m_e m_p} \langle S \cdot I \rangle$$

$$\boxed{g_s = 2}$$

$$\boxed{g_p = 5.5863}$$

$F = S + I$ total spin is conserved (individual spin not)

$$F^2 = S^2 + I^2 + 2S \cdot I \rightarrow S \cdot I = \frac{1}{2}(F^2 - S^2 - I^2)$$

($l=0$)

$$\text{eigenvalues of } S \cdot I : \frac{\hbar^2}{2} [F(F+1) - S(S+1) - I(I+1)]$$

$$\langle \delta^3(r) \rangle = \frac{1}{\pi} \left(\frac{me\alpha c}{n\hbar} \right)^3$$

$$E_{nf}^{(1)} = \frac{2}{3\epsilon_0 c^2} \frac{g_s g_p e^2}{4m_e m_p} \frac{\hbar^2}{2} [F(F+1) - S(S+1) - I(I+1)] \cdot \frac{1}{\pi} \left(\frac{me\alpha c}{n\hbar} \right)^3$$

$$= \frac{g_p e^2 \hbar^2}{m_e m_p 3\epsilon_0 c^2} \cdot \frac{1}{2\pi} \cdot \frac{me^3 c^3}{n^3 \hbar^3} \cdot \frac{e^6}{64\pi^3 \epsilon_0^3 \hbar^3 c^3} [F(F+1) - S(S+1) - I(I+1)]$$

$$= \frac{me e^4 c^4}{2n^2 \cdot 16\pi^2 \epsilon_0^2 \hbar^2 c^2} \cdot \frac{e^4}{16\pi^2 \epsilon_0^2 \hbar^2 c^2} g_p \frac{4}{3} \frac{me}{m_p n} [F(F+1) - S(S+1) - I(I+1)]$$

$$= |E_n^{(0)}| \alpha^2 \frac{4}{3n} g_p \frac{me}{m_p} [F(F+1) - S(S+1) - I(I+1)]$$

b) $n=1, l=0$

spin of electron = $\frac{1}{2}$ $S = \frac{1}{2}$

spin of proton = $\frac{1}{2}$ $I = \frac{1}{2}$

$F = S + I \rightarrow F = 0, 1$

$F=0$ SINGLET

DEGENERACY = 1

$F=1$ TRIPLET

DEGENERACY = 3

c) $n=2$

$$E_f^{(1)} = E_2^{(0)} \left(\frac{\alpha}{2} \right)^2 \left(\frac{2}{2+1/2} - \frac{3}{4} \right) \text{ FINE STRUCTURE } n=2$$

$n=2$ $l = \begin{matrix} 0 \\ 1 \end{matrix}$ $J = L + S$ $S = \frac{1}{2}$ $l=0 \rightarrow J = \frac{1}{2}$

$l=1 \rightarrow J = \frac{1}{2}, \frac{3}{2}$

$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx 0.0073$

$E_2^{(0)} = -\frac{me c^2}{8} \alpha^2 \approx -5.46 \times 10^{-5} \text{ J}$

$J = \frac{1}{2}$ $E_f^{(1)} = E_2^{(0)} \left(\frac{\alpha}{2} \right)^2 \cdot \frac{5}{4} \approx -5.7 \times 10^{-5} \text{ eV}$

$J = \frac{3}{2}$ $E_f^{(1)} = E_2^{(0)} \left(\frac{\alpha}{2} \right)^2 \cdot \frac{1}{4} \approx -1.14 \times 10^{-5} \text{ eV}$

Hyperfine structure $n=1$

$$E_{hf}^{(1)} = |E_1^{(0)}| \frac{4\alpha^2}{3} \rho_p \frac{m_e}{m_p} [F(F+1) - J(J+1) - I(I+1)]$$

$$E_{hf}^{(1)} = |E_1^{(0)}| \frac{4\alpha^2}{3} \cdot 5.5883 \frac{m_e}{m_p} \left[-\frac{1}{2} \left(\frac{3}{2} \right) - \frac{1}{2} \left(\frac{3}{2} \right) \right]$$

$$E_1^{(0)} = -\frac{m_e \alpha^2 c^2}{2} \approx -2.18 \times 10^{-18} \text{ J}$$

$$E_{hf}^{(1)} \approx 2.18 \times 10^{-18} \cdot \frac{4}{3} \left(-\frac{3}{2} \right) \cdot 5.5883 \cdot \frac{m_e}{m_p} \cdot \alpha^2 \approx -7 \times 10^{-25} \text{ J}$$

$$E_{hf}^{(1)} \approx -4.4 \times 10^{-6} \text{ eV}$$

we found that the hyperfine splitting at $n=1$ is smaller than the fine splitting at $n=2$

order of magnitude of the ratio fine structure / hyperfine structure:

$$\frac{\alpha^4 m c^2}{\frac{m_e}{m_p} \alpha^4 m^2} \approx \frac{m_p}{m_e}$$

while the fine structure is due to the spin-orbit coupling, the hyperfine structure is due to weaker interaction, that is the magnetic interaction between the dipole moments of the electron and the nucleus.

Ex 6

$$W_z \approx \frac{eS_z B}{m_e}$$

$$W_{hf} + W_z = W_{total}$$

a) weak B field

if we have a weak B field the hyperfine term dominates, we deal with states of definite F and the Zeeman interaction can be treated as a perturbation

$$E_z^{(1)} = \langle W_z \rangle = \frac{eB}{m_e} \langle S_z \rangle = \frac{eB}{m_e} \mu_B m_F = \mu_B B m_F$$

GROUND STATE	
$n=1$	$l=0$
	$s = \frac{1}{2}$
	$I = \frac{1}{2}$

the energy is

$$E = E_{hf}^{(1)} + E_z^{(1)} = |E_n^{(0)}| \alpha^2 \frac{4}{3} \rho \frac{m_e}{m_p} [F(F+1) - \frac{3}{2}] + \mu_B B m_F$$

b) strong B field

in this case is the Zeeman term that dominates and the hyperfine structure can be treated as a perturbation.

$$E_{hf}^{(1)} = \langle W_{hf} \rangle$$

now the states are $|m_s m_I\rangle$

$$\langle W_{hf} \rangle = |E_n^{(0)}| \frac{4\alpha^2}{3} \rho \frac{m_e}{m_p} \frac{1}{\alpha^2} \langle S \cdot I \rangle$$

$$\langle S \cdot I \rangle = \langle S_z \rangle \langle I_z \rangle = \alpha^2 m_s m_I$$

the energy is:

$$E = |E_n^{(0)}| \frac{4\alpha^2}{3} \rho^2 \frac{m_e}{m_p} m_s m_I + 2\mu_B B m_s$$

EX 7

a) let Φ_T A TRIAL WAVEFUNCTION
the expected value of the energy is:

$$\langle E \rangle = \langle \Phi_T | H | \Phi_T \rangle = \langle \Phi_T | E | \Phi_T \rangle = E_T \langle \Phi_T | \Phi_T \rangle = E[\Phi_T]$$

expanding $|\Phi_T\rangle = \sum_n a_n |n\rangle$

$$|\Phi_T\rangle = \sum_{n,m=0}^{\infty} a_m^* a_n \langle m | H | n \rangle = \sum_{n,m=0}^{\infty} a_m^* a_n E_n \delta_{mn}$$

$$\sum_{n=0}^{\infty} |a_n|^2 E_n = E_0 \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=0}^{\infty} |a_n|^2 (E_n - E_0) \geq E_0$$

$E[\Phi_T]$ is an upper bound to the exact ground state energy for any trial Φ_T wave function

b) $\Phi_T = c e^{-\alpha r}$ TRIAL FUNCTION

first of all we have to normalize Φ

$$|\Phi_T|^2 = 1 \Rightarrow c^2 \int_0^{2\pi} \int_0^\pi \int_0^\infty e^{-2\alpha r} r^2 \sin\theta dr d\theta d\varphi = 1$$

$$4\pi c^2 \int_0^\infty r^2 e^{-2\alpha r} dr = 1 \Rightarrow 4\pi c^2 \cdot \frac{1}{4\alpha^3} = 1$$

$$c^2 = \frac{\alpha^3}{\pi} \Rightarrow c = \alpha \sqrt{\frac{\alpha}{\pi}} \Rightarrow \boxed{\Phi_T = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r}}$$

$\langle H \rangle = \langle T \rangle + \langle V \rangle$ the expectation value of H is the sum of the expectation values of kinetic energy and potential energy

$$\begin{aligned} \langle T \rangle &= \frac{-\hbar^2}{2m} |c|^2 \cdot 4\pi \int_0^\infty e^{-\alpha r} \frac{d^2}{dr^2} e^{-\alpha r} r^2 dr \\ &= -\frac{\hbar^2}{2m} \frac{\alpha^3}{\pi} 4\pi \int_0^\infty e^{-\alpha r} \alpha^2 e^{-\alpha r} r^2 dr = +\frac{\hbar^2}{2m} \alpha^5 \int_0^\infty r^2 e^{-2\alpha r} dr \\ &= +\frac{\hbar^2}{m} \alpha^5 \cdot 2 \cdot \frac{1}{8\alpha^3} = +\frac{\hbar^2}{2m} \alpha^2 \end{aligned}$$

$$\begin{aligned} \langle V \rangle &= -\frac{\alpha^3}{\pi} \cdot \frac{e^2}{4\pi\epsilon_0} \cdot 4\pi \int_0^\infty e^{-\alpha r} \frac{1}{r} e^{-\alpha r} r^2 dr = -\frac{\alpha^3 e^2}{\pi\epsilon_0} \int_0^\infty r e^{-2\alpha r} dr \\ &= -\frac{\alpha^3 e^2}{\pi\epsilon_0} \cdot \frac{1}{4\alpha^2} = -\frac{\alpha e^2}{4\pi\epsilon_0} \end{aligned}$$

$$\langle H \rangle = -\frac{\hbar^2}{2m} \alpha^2 - \frac{e^2}{4\pi\epsilon_0} \alpha$$

to find α we have to find the minimum of $\langle H \rangle$

$$\frac{d\langle H \rangle}{d\alpha} = 0 \rightarrow +2\alpha \frac{\hbar^2}{2m} - \frac{e^2}{4\pi\epsilon_0} = 0 \rightarrow \alpha = +\frac{e^2 m}{4\pi\epsilon_0 \hbar^2}$$

$$\langle H \rangle_{\min} = +\frac{\hbar^2}{2m} \frac{e^4 m^2}{(4\pi\epsilon_0 \hbar^2)^2} - \frac{e^2}{4\pi\epsilon_0} \frac{e^2 m}{4\pi\epsilon_0 \hbar^2} = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \left(\frac{m}{\hbar^2} - \frac{1}{2} \frac{m}{\hbar^2}\right)$$

$$\langle H \rangle_{\min} = -\frac{1}{2} \frac{m}{\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = E_1 = -13.6 \text{ eV}$$

The variational method gives $\langle H_{\min} \rangle = -13.6 \text{ eV}$, that is different from the value of the book, -11.5 eV , obtained using a different $\Psi = A e^{-br^2}$

$$\int_0^\infty x^n e^{-\frac{x}{a}} = n! a^{n+1}$$

c) If the ground state $|0\rangle$ is known, we can find a trial wavefunction ϕ that is orthogonal to the ground state imposing:

$$\langle \phi | 0 \rangle = 0$$

the minimum of $\langle \phi | H | \phi \rangle$ is an upper bound on the energy E_1 of the first excited state.
Then we can find a trial wavefunction orthogonal to $|0\rangle$ and $|1\rangle$ and find an upper bound to E_2 .

It is also possible sometimes to find trial wavefunction orthogonal to the ground state when this is not known.
In this case we need to use symmetry.

For example consider a potential:

$$V(x) = V(-x) \text{ invariant under parity}$$

and we know that the ground state is even and the first excited state is odd under parity.

and using trial wavefunction odd under parity we know that it's surely orthogonal to the ground state and we can use the variational method to find an upper bound to E_1 .

EX 8

a) At the section 9.1 of the book the coupled set is calculated:

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar} [c_a V_{aa} + c_b V_{ab} e^{-i\omega_0 t}] \\ \dot{c}_b = -\frac{i}{\hbar} [c_b V_{bb} + c_a V_{ba} e^{i\omega_0 t}] \end{cases} \quad \omega_0 = \frac{E_b - E_a}{\hbar} (= \omega_{ba})$$

b) differentiating WRT t:

$$\begin{aligned} \ddot{c}_b &= -\frac{i}{\hbar} [\dot{c}_b V_{bb} + \dot{c}_a V_{ba} e^{i\omega_0 t} + i\omega_0 c_a V_{ba} e^{i\omega_0 t}] \\ &= -\frac{i}{\hbar} [c_b V_{bb} + c_a V_{aa} V_{ba} e^{i\omega_0 t} + c_b V_{ab} V_{ba} + i\omega_0 c_a V_{ba} e^{i\omega_0 t}] \end{aligned}$$

from point a) get c_a and put it in $\ddot{c}_b$:

$$\ddot{c}_b = -\frac{i}{\hbar} (c_b V_{aa} + c_b V_{bb} - c_b V_{aa} V_{bb} + \hbar \omega_0 c_b + \hbar \omega_0 c_b V_{bb} - \frac{i}{\hbar} c_b V_{ab} V_{ba})$$

$$\ddot{c}_b = -\frac{i}{\hbar} c_b (V_{aa} + V_{bb} + \hbar \omega_0) + \frac{i}{\hbar} c_b (V_{aa} V_{bb} + \frac{i}{\hbar} |V_{ba}|^2 - \hbar \omega_0 V_{bb})$$

we need c_b in the form: $c_b = e^{\lambda t}$

$$\lambda^2 e^{\lambda t} + \frac{i}{\hbar} \lambda e^{\lambda t} (V_{aa} + V_{bb} + \hbar \omega_0) - \frac{i}{\hbar} e^{\lambda t} (V_{aa} V_{bb} + \frac{i}{\hbar} |V_{ba}|^2 - \hbar \omega_0 V_{bb}) = 0$$

$$e^{\lambda t} \cdot f(\lambda) = 0$$

$$\text{determinant of } f(\lambda): \Delta = -\frac{1}{\hbar^2} (V_{aa} + V_{bb} + \hbar \omega_0)^2 + \frac{4i}{\hbar} (V_{aa} V_{bb} + \frac{i}{\hbar} |V_{ba}|^2 - \hbar \omega_0 V_{bb})$$

$$\lambda_{\pm} = \frac{-\frac{i}{\hbar} \alpha \pm \sqrt{\Delta}}{2}$$

$$c_b = A e^{\lambda_+ t} + B e^{\lambda_- t}$$

$$c_b = e^{-i(\alpha - \omega_0)t} (M \cos(\beta t) + N \sin(\beta t))$$

$$c_b(0) = \boxed{M=0} \Rightarrow c_b = N e^{-i(\alpha - \omega_0)t} \sin(\beta t)$$

$$c_a = i\hbar \frac{\dot{c}_b - c_b V_{bb}}{V_{ba}} e^{i\omega_0 t}, \quad c_a(0) = 1 \Rightarrow \boxed{N = -i \frac{V_{ba}}{\hbar \beta}}$$

$$\text{where } \alpha = \frac{1}{2\hbar} (V_{aa} + V_{bb} + \hbar \omega_0), \quad \beta = \frac{1}{\hbar} \left[\frac{1}{4} (V_{bb} - V_{aa} + \hbar \omega_0)^2 + |V_{ba}|^2 \right]^{\frac{1}{2}}$$

$$\gamma = \alpha - \frac{V_{aa}}{\hbar}$$

$$\Rightarrow C_b = -i \frac{V_{be}}{\hbar \beta} e^{-i(\alpha - \omega_0)t} \sin \beta t$$

$$C_a = i \frac{1}{\hbar} \frac{C_b - C_b V_{bb}}{V_{be}} e^{i\omega_0 t} = \frac{1}{\hbar} \frac{1}{V_{be}} e^{i\omega_0 t} \left(-\frac{i V_{be}}{\hbar \beta} \right) (-i(\alpha - \omega_0) e^{i(\alpha - \omega_0)t} \sin \beta t + \cos \beta t)$$

$$C_a = e^{i\alpha t} \left(\cos \beta t + \frac{1}{\beta} \sin \beta t \right)$$

c) first order perturbation

$$C_a^1 = 1 - \int_0^t C_a dt = 1 - \frac{i}{\hbar} \int_0^t C_a V_{aa} dt - \frac{i}{\hbar} \int_0^t C_b V_{ab} e^{-i\omega_0 t} dt$$

$$C_a^1 = 1 - \frac{i}{\hbar} C_a V_{aa} t - \frac{i}{\hbar} (-i\omega_0 C_b V_{ab} e^{-i\omega_0 t})$$

$$C_b^1 = -\frac{i}{\hbar} \int_0^t C_b V_{bb} dt - \frac{i}{\hbar} \int_0^t C_a V_{be} e^{i\omega_0 t} dt$$

$$= -\frac{i}{\hbar} C_b V_{bb} t - \frac{i}{\hbar} (i\omega_0 C_a V_{be} e^{i\omega_0 t})$$

$$\begin{cases} C_a^1 = 1 - \frac{i}{\hbar} [C_a V_{aa} t - i\omega_0 C_b V_{ab} e^{i\omega_0 t}] \\ C_b^1 = -\frac{i}{\hbar} [C_b V_{bb} t + i\omega_0 C_a V_{be} e^{i\omega_0 t}] \end{cases}$$

d) $E_b^{(0)} = E_a^{(0)}$ $V_{aa} = V_{bb} = 0$, $\omega_0 = 0$, $\alpha = 0$, $\beta = \frac{1}{\hbar} |V_{ba}|$

$$\begin{cases} \dot{C}_a = -\frac{i}{\hbar} C_b V_{ab} \\ \dot{C}_b = -\frac{i}{\hbar} C_a V_{be} \end{cases}$$

$$\begin{cases} C_a = e^0 (\cos \beta t) = \cos \left(\frac{|V_{be}|}{\hbar} t \right) \\ C_b = -\frac{i}{\hbar} V_{ba} - \frac{1}{\beta} \sin \beta t = -\frac{i}{\hbar} \left(\frac{|V_{be}|}{\hbar} t \right) \end{cases}$$

~~$$\begin{cases} C_a = \frac{i}{\hbar} \left[\int_0^t C_a A dt + \int_0^t C_b A e^{i(\omega - \omega_0)t} dt \right] \\ \quad - \frac{i}{\hbar} C_a A t - \frac{1}{\hbar(\omega - \omega_0)} C_b A + e^{-i(\omega - \omega_0)t} \\ C_b = -\frac{i}{\hbar} C_b A t + \frac{1}{\hbar(\omega - \omega_0)} C_a A + e^{-i(\omega - \omega_0)t} \end{cases}$$~~

f) first order perturbation

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar} \left[c_a A - \frac{i}{\omega - \omega_0} c_b A e^{i(\omega - \omega_0)t} \right] \\ \dot{c}_b = -\frac{i}{\hbar} \left[c_b A^\dagger + \frac{i}{\omega - \omega_0} c_a A^\dagger e^{-i(\omega - \omega_0)t} \right] \end{cases}$$

g) $\epsilon^{(0)} = E_a^{(0)} \quad \omega_0 = \omega$

$$c_a = -\frac{i}{\hbar} \left[c_a A t - \frac{i}{\omega} c_b A e^{i\omega t} \right]$$

$$c_b = -\frac{i}{\hbar} \left[c_b A^\dagger t + \frac{i}{\omega} c_a A^\dagger e^{-i\omega t} \right]$$

in these case the coefficient depend on ω

e) $V(t) = A e^{i\omega t} + A^\dagger e^{-i\omega t}$

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar} \left[c_a A + c_b A e^{i(\omega - \omega_0)t} \right] \\ \dot{c}_b = -\frac{i}{\hbar} \left[c_b A^\dagger + c_a A^\dagger e^{-i(\omega - \omega_0)t} \right] \end{cases}$$

$$\begin{cases} c_a = -\frac{i}{\hbar} \left[\int_0^t c_a A dt + \int_0^t c_b A e^{i(\omega - \omega_0)t} dt \right] \quad A \text{ is time independent} \\ = -\frac{i}{\hbar} c_a A t - \frac{1}{\hbar(\omega - \omega_0)} c_b A e^{i(\omega - \omega_0)t} \\ c_b = \frac{i}{\hbar} \left[c_b A^\dagger t + \frac{1}{\hbar(\omega - \omega_0)} c_a A^\dagger e^{-i(\omega - \omega_0)t} \right] \end{cases}$$