Practice Session — Physics Laboratory: Skills

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Give motivations and/or derivations for your answers.
Using the integral table of a Gaussian and the table for the cumulative χ^2 is allowed.

1. **Observing Decays**
The decay of a sample of radioactive material is measured and an *average* rate of 3.5 decays per second is observed.

Calculate what the probability P is of observing exactly 5 decays in 1 second, given the following:

$$f(k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

Solution: **Calculate** what the probability P is of observing exactly 5 decays in 1 second.
Probability $P = f(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ with $\lambda = 3.5$ and $k = 5$ so $P = f(k) = \frac{3.5^5}{5!}e^{-3.5}$ so $\boxed{P = 13\%}$.

2. **Measuring Brightness**
We are measuring the brightness of a faint star, i.e. the rate of incoming photons, which we calculate to be $N = 4 \cdot 10^8$ photons under one second of exposure time.

(a) **Compute** the uncertainty in the number of detected photons, assuming that it is a Poisson random variable!

(b) **Reason** why we still make long exposures with telescopes, even though the uncertainty in the number of detected photons is increasing with exposure time! *Hint: Think of the relative uncertainty in brightness.*

Solution:

(a) The uncertainty in a Poisson process is $\sqrt{N}$, so that the uncertainty in the number of detections is

$$\boxed{\sqrt{N} = 2 \cdot 10^4 \text{ photons}}$$

(b) The relative uncertainty in the photon counts is given by

$$\delta N = \frac{\sigma_N}{|N|} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\lambda \cdot t_{\text{exp}}}}$$

Clearly, as we increase t_{exp}, the ***relative* uncertainty diminishes**. Also, one can see that a brighter star (with a greater rate parameter λ) has a less uncertain brightness.

3. **Proper Accuracy**
Rewrite the following results using the correct notation (proper accuracy and number of significant digits)

(a) $f = 31.41$ Hz ± 0.5 Hz

(b) $F = 3.141$ kdyn ± 59 dyn

(c) $S = 3.1$ μJy ± 41 nJy

(d) $\Omega = 314.159$ sr ± 2 msr

(e) $E = 3141$ kerg ± 0.592 Merg

Solution:

(a) $f = 31.41$ Hz ± 0.5 Hz
$\boxed{f = 31.4 \pm 0.5 \text{ Hz}}$

(b) $F = 3.14$ kdyn ± 59 dyn
$\boxed{F = 3.14 \pm 0.06 \text{ kdyn}}$

(c) $S = 3.1$ μJy ± 41 nJy
$\boxed{S = 3.10 \pm 0.05\ \mu\text{Jy}}$

(d) $\Omega = 314.159$ sr ± 2 msr
$\boxed{\Omega = 314159 \pm 2 \text{ msr}}$

(e) $E = 3141$ kerg ± 0.592 Merg
$\boxed{E = 3.1 \pm 0.6 \text{ Merg}}$

4. **Escaping With Your Spaceship**
The escape velocity v is the minimum velocity required to leave a planet with your spaceship and is such that it overcomes the pull of gravity. It is given by:

$$v = \sqrt{\frac{2Gm}{r}}$$

with escape velocity v, planet radius r, planet mass m, and the gravitational constant G.

(a) **Give** the (SI) units of G. Show at least one intermediate step.

(b) **Calculate** the relative error and the absolute error in v given that a space shuttle with some velocity v tries to escape a planet of $m = (1234.567 \pm 0.891) \cdot 10^{20}$ kg, with $G = 6.67430 \cdot 10^{-11}$ (in the appropriate units) and $r = (4534.234 \pm 0.762) \cdot 10^2$ m.

(c) **Write** the final result in the correct notation $v = ... \pm ....$

Solution:

(a) **Give** the (SI) units of G. Show at least one intermediate step.

$$\frac{\text{m}}{\text{s}} = \sqrt{\frac{[G]\text{kg}}{\text{m}}} \quad \Rightarrow \quad \frac{\text{m}^2}{\text{s}^2} = \frac{[G]\text{kg}}{\text{m}} \quad \Rightarrow \quad \boxed{[G] = \frac{\text{m}^2}{\text{s}^2}\frac{\text{m}}{\text{kg}} = \frac{\text{m}^3}{\text{s}^2\ \text{kg}}}$$

(b) **Calculate** the relative error and the absolute error in v given that a space shuttle with some velocity v tries to escape a planet of $m = (1234.567 \pm 0.891) \cdot 10^{20}$ kg, with $G = 6.67430 \cdot 10^{-11}$ and $r = (4534.234 \pm 0.762) \cdot 10^2$ m.

$$v = \sqrt{\frac{2Gm}{r}} = \boxed{6.029 \cdot 10^3 \text{ m/s}}$$

substitute $z = 2Gm/r$, then $v = \sqrt{z}$ and

$$\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta r}{r}\right)^2 = 5.491 \cdot 10^{-7}$$

so

$$\left(\frac{\Delta v}{v}\right) = \frac{1}{2}\left(\frac{\Delta z}{z}\right) \Rightarrow \left(\frac{\Delta v}{v}\right)^2 = \left(\frac{1}{2}\right)^2\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{1}{2}\right)^2\left[\left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta r}{r}\right)^2\right]$$

$\Rightarrow \frac{\Delta v}{v} = \boxed{3.705 \cdot 10^{-4}} \Rightarrow \boxed{\Delta v = 2.234 \text{ m/s}}$

(c) **Write** the final result in the correct notation $v = ... \pm ....$
$\boxed{v = (6.029 \pm 0.003) \cdot 10^3 \text{ m/s}}$

5. **Rowing Competition**
In lightweight rowing for men, the average weight of the team members is not allowed to be higher than 70 kg. The weight of each team member is not allowed to be more than 72.5 kg. *Robert* (68.1 ± 0.3 kg) and *Hans* (72.1 ± 0.4 kg) want to participate in a lightweight rowing competition. They will measure their weight.

(a) **Estimate** the probability that Hans is too heavy to participate in the competition.

Hans weight is also determined with another scale, giving 72.4 ± 0.3 kg.

(b) **Calculate** the weighted average and weighted error of the weights for Hans.

(c) **Estimate** the probability that Robert and Hans are not allowed to participate on the basis of average weight (using the weights given at the start of the question) .

Robert and Hans decide to take part in the competition. The crew of the competition knows nothing about statistics and random errors and therefore performs only one measurement of each of the team members' weights, finding the following results: Robert 67.9 kg, Hans 72.3 kg. By calculation of the average weight the crew decides to disqualify the team.

(d) **Calculate** the average weight the crew finds.

Robert is not to be caught for one hole. He decides to interchange his everyday shoes for socks and sandals. Assume that on average socks and sandals are 0.3 kg lighter than normal shoes.

(e) **Reason** whether this changing of shoes is enough to be allowed to participate in the competition.

Solution:

(a) **Estimate** the probability that Hans is too heavy to participate in the competition.
Above one standard deviation, approximately $\boxed{15.8\%}$.

(b) **Calculate** the weighted average and weighted error of the weights.

$$w_1 = \frac{1}{s_1^2} = \frac{1}{0.3^2} = 11.11 \qquad w_2 = \frac{1}{s_2^2} = \frac{1}{0.4^2} = 6.25$$
$$M = \frac{w_1 M_1 + w_2 M_2}{w_1 + w_2} = \frac{11.11 \cdot 68.1 + 6.25 \cdot 72.1}{11.11 + 6.25} = 72.29 \text{ kg} \approx \boxed{72.3 \text{ kg}}$$
$$\frac{1}{s_M^2} = \frac{1}{s_1^2} + \frac{1}{s_2^2} \qquad \boxed{s_M = 0.24 \text{ kg} \approx 0.3 \text{ kg}}$$

(c) **Estimate** the probability that Robert and Hans are not allowed to participate on the basis of average weight.
The (weighted) average of Robert and Hans: $\mu \approx 69.5$ kg, $\sigma \approx 0.3$ kg
Calculate $z = |(x - \mu)/\sigma| \approx 1.7$. This yields $F \approx 0.4554$, so the probability is $0.5 - 0.4554 \approx 0.045 = \boxed{4.5\%}$.

Robert and Hans decide to take part in the competition. The crew of the competition knows nothing about statistics and random errors and therefore performs only one measurement of each of the team members' weights, finding the following results: Robert 67.9 kg, Hans 72.3 kg. By calculation of the average weight the crew decides to disqualify the team.

(d) **Calculate** the average obtained by the crew.

$$M_{\text{average}} = \frac{1}{N}\sum_{i=1}^{N} M_i = \frac{67.9 + 72.3}{2} = \boxed{70.1 \text{ kg}}$$

Robert is not to be caught for one hole. He decides to interchange his everyday shoes for sandals. Assume that on average sandals are 0.3 kg lighter than normal shoes.

(e) **Reason** whether this changing of shoes is enough to be allowed to participate in the competition.
One could say that Robert's weight is now 67.6 kg, resulting in an average weight of 69.95 kg, but this is not true since Robert's everyday shoes are already sandals.

6. **The Mass Of The Milky Way**
Two students have bought *Sint Maarten* candy, but no children came to their home to sing, so they are left with a bag full of mini Milky Way bars. They start measuring the mass of the bars, to see whether all bars are equally heavy. The results are as follows: $m_1 = 16.6$ g, $m_2 = 16.8$ g, $m_3 = 16.5$ g, $m_4 = 16.0$ g, $m_5 = 16.9$ g

(a) **Calculate** the average mass $\bar{m}$.

(b) **Calculate** the best estimate for the standard deviation in these measurements.

(c) **Determine** the error s_m, the standard error in the mean.

(d) **Reason** how many measurements should be taken in order to reduce the error in these measurements by a factor 3.

Solution:

(a) **Calculate** the average mass $\bar{m}$.

$$\bar{m} = \frac{1}{N}\sum_{i=1}^{N} m_i = \frac{1}{5}(16.6 + 16.8 + 16.5 + 16.0 + 16.9) = \boxed{16.56 \text{ g}}$$

(b) **Calculate** the best estimate for the standard deviation in these measurements.

$$s_m = \sqrt{\frac{\sum_{i=1}^{N}(m_i - m)^2}{N-1}} = \sqrt{\frac{(0.04)^2 + (0.24)^2 + (-0.06)^2 + (-0.56)^2 + (0.34)^2}{4}} \approx 0.3507 \text{ g} = \boxed{0.4 \text{ g}}$$

(c) **Determine** the error s_m, the standard error in the mean.

$$s_{\bar{m}} = \frac{s_m}{\sqrt{N}} \approx \frac{0.3507 \text{ g}}{\sqrt{5}} \approx 0.16 \text{ g} = \boxed{0.2 \text{ g}}$$

(d) **Reason** how many more measurements should be taken to reduce the error in these measurements by a factor 3.
The error in the mean decreases by a factor of $\sqrt{N}$, so in order for the error to be 3 times as small, the number of measurements should be $3^2 = 9$ times as large. This yields $5 \cdot 9 = \boxed{45}$ measurements.

7. **Physics Lab Grades**

q	$n \pm \Delta n$
1.00	10 ± 2
2.00	25 ± 2
3.00	36 ± 2
4.00	39 ± 2

A series of 4 observations is given in the table above, here q is the grade of a student on their Physics Lab 1 exam and n is the number of students with this grade. The error in q is negligible. A straight line $n = a \cdot q + b$ is fitted to these observations. The following formulae are given.

$$a = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N\sum x_i^2 - (\sum x_i)^2}$$

$$\Delta a = \sqrt{\left(\frac{1}{\sum x_i^2 - N\bar{x}^2}\right)\frac{\sum r_i^2}{N-2}}$$

$$\Delta b = \sqrt{\left(\frac{1}{N} + \frac{\bar{x}^2}{\sum x_i^2 - N\bar{x}^2}\right)\frac{\sum r_i^2}{N-2}}$$

(a) **Calculate** the best estimate for a and b using the method of least squares.

(b) **Calculate** the uncertainties in a and b.

(c) **Calculate** χ^2, and **argue** whether the linear fit is acceptable on the 10-90% probability level.

Solution:

(a) **Calculate** the best estimate for a and b using the method of least squares.
$N = 4$, $\sum q_i = 10$, $\sum n_i = 110$, $\sum q_i^2 = 30$, $\sum q_i n_i = 324$
$\Rightarrow \boxed{a = 9.8}$
and $\bar{q} = 2.5$, $\bar{n} = 27.5 \Rightarrow \boxed{b = \bar{n} - a\bar{q} = 3.0}$.

(b) **Calculate** the uncertainties in a and b.
$r_i = n_i - f(q_i)$, $\sum r_i^2 = (10 - 12.8)^2 + (25 - 22.6)^2 + (36 - 32.4)^2 + (39 - 42.2)^2 = 36.8$
$\Rightarrow (\Delta a)^2 = (30 - 4 \cdot 2.5^2)^{-1}(36.8/2) = 3.68 \Rightarrow \boxed{\Delta a = 2.0}$
and $(\Delta b)^2 = 27.6 \Rightarrow \boxed{\Delta b = 5.3}$

(c) **Calculate** χ^2, and **argue** whether the linear fit is acceptable on the 10-90% probability level.

By definition, $\chi^2 = \sum_{i=1}^{N}\left(\frac{y_i - f(x_i)}{\Delta y_i}\right)^2$ which, for this problem, reads

$$\chi^2 = \sum_{i=1}^{4}\left(\frac{n_i - aq_i - b}{\Delta n}\right)^2 = \boxed{9.2}$$

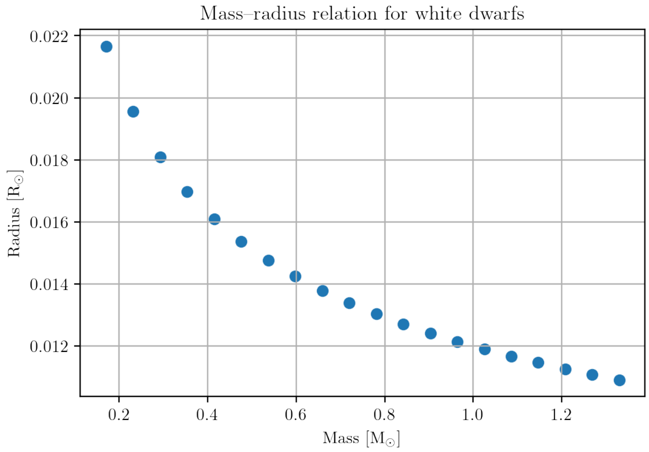
We are fitting two parameters, to the degrees of freedom are $\nu = N - n_{\text{param}} = 2$, for which the threshold values for the cumulative χ^2 are 0.211 and 4.605 for 10% and 90%, respectively. As χ^2 is significantly greater than the upper limit, the data is underfitted, the linear fit is **not acceptable**.

8. **White Dwarfs**
Interestingly, white dwarf stars with smaller radii have greater masses. The relation between these variables can be cast in the form

$$R = kM^\beta$$

where M and R is measured in solar masses and radii, respectively.

Using the graph below, **estimate** the constants k and β! What is the dimension of k? You do not have to estimate uncertainties.



Solution: We choose two nicely placed data points with easily readable coordinates, (x_1, y_1) and (x_2, y_2).

We know $y_1 = kx_1^\beta$ and $y_2 = kx_2^\beta$, so dividing the equations yields

$$\frac{y_2}{y_2} = \left(\frac{x_2}{x_1}\right)^\beta \implies \ln(y_2/y_1) = \beta \ln(x_2/x_1) \implies \beta = \frac{\ln(y_2/y_1)}{\ln(x_2/x_1)}$$

Choosing the fifth point with $(M, R) = (0.4\ \mathrm{M}_\odot, 0.016\ \mathrm{R}_\odot)$ and an interpolated point at $(M, R) = (1.0\ \mathrm{M}_\odot, 0.012\ \mathrm{R}_\odot)$, using the relation for β gives:

$$\boxed{\beta = -0.314}$$

which is close to the theoretical value of $\beta = -1/3$.

Finally, using the second data point, we have

$$R = kM^\beta \implies k = RM^{-\beta} \implies \boxed{k = 0.012}$$

9. **Unit Conversion**
Convert the following data into the desired units! It is given that $1\ \mathrm{pc} = 3.086 \cdot 10^{16}\ \mathrm{m}$.

Value	Convert to:
$5.67 \cdot 10^{-8}\ \mathrm{W\ m^{-2}K^{-4}}$	$\mathrm{mW\ cm^{-2}\ nK^{-4}}$
$1.055 \cdot 10^{-34}\ \mathrm{Js}$	$\mu\mathrm{g\ km^2\ h^{-1}}$
$70\ \mathrm{km\ s^{-1}Mpc^{-1}}$	year^{-1}

Solution: We are asked to convert the following quantities into the specified units. It is given that

$$1\ \text{pc} = 3.086 \times 10^{16}\ \text{m}.$$

(1) $5.67 \times 10^{-8}\ \text{W}\,\text{m}^{-2}\,\text{K}^{-4} \to \text{mW}\,\text{cm}^{-2}\,\text{nK}^{-4}$

We perform the conversions step by step:

$$1\ \text{W} = 10^3\ \text{mW}, \quad 1\ \text{m}^2 = 10^4\ \text{cm}^2 \Rightarrow 1\ \text{m}^{-2} = 10^{-4}\ \text{cm}^{-2},$$

$$1\ \text{K} = 10^9\ \text{nK} \Rightarrow 1\ \text{K}^{-4} = 10^{-36}\ \text{nK}^{-4}.$$

Combining all factors:

$$\begin{aligned} &5.67 \times 10^{-8}\ \text{W}\,\text{m}^{-2}\,\text{K}^{-4} \times 10^3 \times 10^{-4} \times 10^{-36} \\ &= \boxed{5.67 \times 10^{-45}\ \text{mW}\,\text{cm}^{-2}\,\text{nK}^{-4}} \end{aligned}$$

(2) $1.055 \times 10^{-34}\ \text{J}\,\text{s} \to \mu\text{g}\,\text{km}^2\,\text{h}^{-1}$

Since $1\ \text{J} = 1\ \text{kg}\,\text{m}^2\,\text{s}^{-2}$, we have

$$1.055 \times 10^{-34}\ \text{J}\,\text{s} = 1.055 \times 10^{-34}\ \text{kg}\,\text{m}^2\,\text{s}^{-1}.$$

Now,

$$1\ \text{kg} = 10^9\ \mu\text{g}, \quad 1\ \text{m}^2 = 10^{-6}\ \text{km}^2, \quad 1\ \text{s}^{-1} = 3600\ \text{h}^{-1}.$$

Thus,

$$\begin{aligned} &1.055 \times 10^{-34}\ \text{kg}\,\text{m}^2\,\text{s}^{-1} \times 10^9 \times 10^{-6} \times 3600 \\ &= \boxed{3.8 \times 10^{-28}\ \mu\text{g}\,\text{km}^2\,\text{h}^{-1}} \end{aligned}$$

(3) $70\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} \to \text{year}^{-1}$

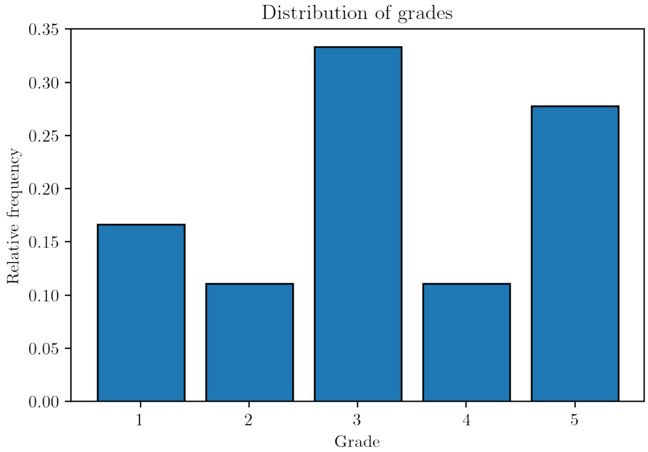
Using $1\ \text{Mpc} = 10^6\ \text{pc} = 3.086 \times 10^{22}\ \text{m}$, we get

$$\begin{aligned} 70\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} &= 70 \times 10^3\ \text{m}\,\text{s}^{-1} \times (3.086 \times 10^{22}\ \text{m})^{-1} \\ &= 2.27 \times 10^{-18}\ \text{s}^{-1}. \end{aligned}$$

Since $1\ \text{year} = 3.156 \times 10^7\ \text{s}$, we find

$$2.27 \times 10^{-18}\ \text{s}^{-1} \times 3.156 \times 10^7\ \text{s/year} = \boxed{7.2 \times 10^{-11}\ \text{year}^{-1}}$$

10. **Grade Distribution**
    Knowing that 18 students have taken a test, estimate the number of them who have scored a 2 or lower, using the figure below.



Solution: We see that the fraction of students with a 2 is about 0.11. Computing the corresponding 11% of 18 students gives 1.98, which is approximately 2 (as, fortunately, the number of students is an integer).

Similarly for grade 1, we can read from the graph that the fraction of students with this grade is approximately 0.17, meaning that $18 \cdot 0.17 \approx 3$ students obtained this mark.

Therefore, a total of **5 students** have obtained a grade of 2 or lower.