Practice session: Cosmology

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Don't forget to give motivations and/or derivations for your answers!

1. Cosmic dynamics can be described by three equations: the Friedmann equation, the acceleration equation and the fluid equation, which are respectively given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\epsilon_m + \epsilon_r) - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3}$$
$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0,$$

where $P = \omega\epsilon$ is the equation of state between pressure P and energy density ϵ.

(a) Use the fluid equation for a flat universe with no cosmological constant to show that the energy density evolves with the expansion factor as

$$\epsilon = \epsilon_0 a^{-3(1+\omega)},$$

with $\omega \neq 1$.

(b) Use the Friedmann equation and the answer in (a) to infer that, for the same type of universe, the expansion factor evolves in time as

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}.$$

(c) Use this to show that the Hubble parameter evolves as

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+\omega)} t_0^{-1}.$$

What is the physical meaning of t_0?

(d) Calculate the energy density evolution for a matter dominated universe ($\omega = 0$), and for a radiation dominated universe ($\omega = \frac{1}{3}$). For both universes, physically explain the dependence on a.

Solution:

(a) Rewriting the fluid equation gives

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + \omega\epsilon) = 0 \implies \frac{\dot{\epsilon}}{\epsilon} = -3\frac{\dot{a}}{a}(1+\omega).$$

Dividing out $\mathrm{d}t$ and integrating both sides, we obtain

$$\int_{\epsilon_0}^{\epsilon(a)} \frac{\mathrm{d}\epsilon}{\epsilon} = -3(1+\omega)\int_{a_0}^{a} \frac{\mathrm{d}a}{a} \implies \ln\left(\frac{\epsilon(a)}{\epsilon_0}\right) = -3(1+\omega)\ln\left(\frac{a}{a_0}\right).$$

Taking out the logarithms and setting $a_0 = 1$ we see that our answer becomes

$$\frac{\epsilon(a)}{\epsilon_0} = a^{-3(1+\omega)},$$

which is equivalent to the equation from the question.

(b) The Friedmann equation for a flat universe with no cosmological constant and with the energy density calculated in (a) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon_0 a^{-3(1+\omega)}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_0 a^{-(1+3\omega)}$$

We make the educated guess that $a \propto t^q$, which means:

$$\begin{aligned} t^{2q-2} &\propto t^{-q(1+3\omega)} \\ 2q-2 &= -q(1+3\omega) \\ q &= \frac{2}{(3(1+\omega))} \quad \text{for } \omega \neq -1 \end{aligned}$$

Substituting the found expression for q into our educated guess gives the answer:

$$a \propto t^q = t^{\frac{2}{3(1+\omega)}}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}$$

(c) Filling in the expansion factor from the previous question we obtain

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \left.\frac{\frac{2}{3(1+\omega)}\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}-1}\cdot\frac{1}{t_0}}{\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}}\right|_{t=t_0} = \left.\frac{2}{3(1+\omega)}\cdot\frac{t_0}{t}\cdot\frac{1}{t_0}\right|_{t=t_0} = \frac{2}{3(1+\omega)}t_0^{-1}.$$

t_0 is the age of the universe, proportional to the Hubble time.

(d) For matter we get

$$\epsilon = \epsilon_0 a^{-3}.$$

The $\frac{1}{a^3}$ is simply the consequence of an expanding volume. For radiation the energy density becomes

$$\epsilon = \epsilon_0 a^{-4},$$

where the extra factor $\frac{1}{a}$ is the consequence of redshift.

2. Explain the following cosmological concepts:

   (a) Cosmological principle
   (b) Inflation
   (c) Copernican principle
   (d) Weak/Strong Anthropic principle
   (e) Metric

Solution:

(a) The density distribution in the universe is homogeneous and isotropic

(b) A state of exponential expansion in the very early universe

(c) The idea that we are not privileged observers in any way

(d)
- Weak: The fact that we can observe the universe allows us to deduce properties of it.
- Strong: The universe has to allow life to be formed at some point.

(e) An object that defines distances in spacetime

3. Olber's Paradox

   (a) State Olber's paradox and give a few resolutions for the problem.

Solution:

(a) Given that the universe is homogeneous and isotropic, every line of sight should eventually end up in a star. However, the sky is (mostly) dark. Possible resolution are that the universe has a finite age, so light from very far away stars hasn't reached us yet. Also, the expansion of the universe will redshift light down the spectrum, causing it to become dark at long distances. Think about the CMB.

4. Distances in the Universe.
   There are several types of distance measures available, a few of these are the so called proper distance (d_p), luminosity distance (d_L) and the angular-diameter distance (d_A).

   (a) What is the difference between the proper distance d_p and the co-moving distance r found in the RW-metric, which is the physical distance?
   (b) Briefly explain the difference between these three distance measures.
   (c) What is the effect of the curvature of space time on the luminosity distance?

(d) How does the expansion of space affect the luminosity distance?

(e) Given these considerations, write down an expression for the luminosity distance.

(f) Do the same steps for the angular-diameter distance; what is the relation between d_A and d_L?

(g) For a model universe (which is not a Λ cosmology) the angular-diameter distance reaches a maximum after which objects start growing in size with increasing redshift, explain this phenomenon.

Solution:

(a) The co-moving distance is the distance between objects if the universe was non-evolving (with constant expansion factor), the proper distance is the physical distance between objects in an evolving universe.

(b) The proper distance is a measure of the physical distance between the observer and the source. The luminosity distance is the distance that can be inferred from knowing the true luminosity of an object and measuring the flux received. The angular-diameter distance is the distance obtained by considering a standard yardstick of length l and dividing by the angle it subtends on the sky (valid for small angles only). If space is static and Euclidean: $d_p = d_L = d_A$. The mathematical expression of the distance are given by

$$d_p = c\int_0^t \frac{dt}{a(t)} \tag{1}$$

$$d_L = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}} \tag{2}$$

$$d_A = \frac{l}{\delta\theta} \tag{3}$$

(c) The curvature of space time causes the proper surface area of the sphere over which photons are spread to change. In general

$$A_p(t_0) = 4\pi S_\kappa(r)^2,$$

where

$$S_\kappa = \begin{cases} R_0 \sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0 \sinh(r/R_0) & \kappa = -1 \end{cases}$$

(d) The expansion of the Universe affects the luminosity distance twofold. Firstly, the energy of the photons is lowered due the the redshift caused by the expansion. Secondly, the rate of incoming photons is lowered because the proper distance between the photons is stretched. This causes the flux received to be lowered by a factor $(1+z)^{-2}$

(e) From the considerations we know that the flux received will be

$$F = \frac{L}{4\pi S_\kappa(r)^2(1+z)^2}.$$

From Equation 2 we know that

$$d_L = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}}$$

thus

$$d_L = S_\kappa(r)(1+z)$$

(f) The distance between the two ends of the yardstick is given by

$$ds = a(t_e)S_\kappa(r)\delta\theta.$$

The factor $a(t_e)$ is due to the expansion of the Universe and the factor $S_\kappa(r)$ takes into account the geometry. For a standard yardstick we can set $ds = l$. Combining this with

$$d_A = \frac{l}{\delta\theta}$$

we find that

$$d_A = \frac{S_\kappa(r)}{1+z}.$$

Therefore

$$d_A = \frac{d_L}{(1+z)^2}.$$

(g) The angle made on the sky by an object with a certain length scales linearly with the distance between you and the object. The expansion of the universe will make it look like certain lengths are longer than they seem. For smaller distances we can ignore expansion. Fast expansion at earlier times can cause high redshift objects to be stretched a lot (canceling the optical effect) while closer objects that have expanded less will appear smaller as they are mostly subject to the optical decrease in size. It is important to note that the objects don't actually change is size, just the relative locations of the photons it emitted.

5. Big Bang Problems
   (a) There are several fundamental problems in the standard Big Bang model, state three of them.
   (b) Give a possible solution for at least two of the problems you've listed.

Solution:

(a) There is the horizon problem, the monopole problem and the flatness problem.
The horizon problem states that the CMB has the same properties everywhere, even though different parts of the CMB should have been outside of each other's particle horizons at the time of recombination. Therefore, there is no reason why the CMB should have the same properties everywhere.

The monopole problem states that there should be magnetic monopoles present in the universe due to topological defects, but none have been found so far.
The flatness problem states that it is extremely unlikely that the universe is close to the critical density at this time, because the deviation from the critical density increases exponentially with time. It therefore narrows down the possible values for the density in the very early universe to an incredibly small range. Since there is no real reason why the universe should have a value in this range, it must mean that we're incredibly lucky.

(b) All problems can be solved in some way by introducing the concept of inflation.
The horizon problem is solved by inflation since inflation allows all parts of the universe to be causally connected in the very early universe.
The monopole problem is solved by inflation since inflation causes the expected distances between monopoles to expand so much that it makes sense that we haven't observed any yet.
Finally, the flatness problem is solved by inflation since inflation forces the density of the universe towards the critical density. This is visible in the Friedmann equation.

6. Event Horizon
   (a) What is the difference between an event horizon and a particle horizon?
   (b) Bulbasaur, Squirtle and Charmander decide to use their cosmological knowledge to try and send information from beyond others event horizon. Bulbasaur and Squirtle are not within each others horizons, Charmander however, is located in both. Therefore, Bulbasaur had the clever idea to send a signal to Charmander, who would then send it to Squirtle. Explain to Bulbasaur why this will not work and why he is thus clearly the worst starter.
   (c) Compute the particle horizon at a given time t for the cosmologies given in Table 1
   (d) Three of these cosmologies are also known under certain names. Which three are these and what are their names.

	Ω_Λ	Ω_M	Ω_R
Cosmology 1	0	0	0
Cosmology 2	1	0	0
Cosmology 3	0	1	0
Cosmology 4	0	0	1

Table 1: Different Cosmologies

Solution:

(a) An event horizon indicates how far you can send information into the future. A particle horizon indicates how far away from you signals can have come from the past.

(b) At the time Bulbasaur's signal reaches Charmander, Squirtle will have left Charmander's event horizon (by definition). Therefore Charmander can not pass the signal on to Squirtle.

(c) The equation required to calculate the particle horizons is given by

$$d_p(t) = a(t) \int_0^t \frac{c\,dt'}{a(t')}$$

For cosmology 1, we know that $a(t) \propto t$. Therefore, we find that the horizon distance is infinite.
For cosmology 2, we know that $a(t) \propto e^{Ht}$. Note that in this case the universe actually is infinitely old, so the lower boundary in the equation for the particle horizon distance isn't 0 but $-\infty$. This causes the particle horizon to also be infinite.
For cosmology 3, we know that $a(t) \propto t^{2/3}$. Therefore, we can calculate the particle horizon distance as

$$\begin{aligned} d_p(t) &= c\,t^{2/3} \int_0^t \frac{dt'}{t'^{2/3}} \\ &= c\,t^{2/3}(3\sqrt[3]{t} - 3\sqrt[3]{0}) \\ &= 3ct \end{aligned}$$

For cosmology 4, we know that $a(t) \propto t^{1/2}$. Therefore, we can calculate the particle horizon distance as

$$\begin{aligned} d_p(t) &= c\,t^{1/2} \int_0^t \frac{dt'}{t'^{1/2}} \\ &= c\,t^{1/2}(2\sqrt[2]{t} - 2\sqrt[2]{0}) \\ &= 2ct \end{aligned}$$

(d)
- Cosmology 1 is known as a Milne universe
- Cosmology 2 is known as a de Sitter universe
- Cosmology 3 is known as an Einstein-de Sitter universe
- Cosmology 4 has no other name than "radiation dominated universe"

7. Nucleosynthesis
   (a) Explain the significance of the ~ 1 MeV energy scale in the early universe.
   (b) Explain how neutrons survive the hot big bang, even though free neutrons decay in 15 minutes.

Solution:

(a) There are many processes that are related to an energy scale of 1 MeV. For instance, the mass difference between a proton and a neutron is approximately 1 MeV. Also, the binding energy of the hydrogen molecule is about 1 MeV. Finally, it is also the energy scale associated with the neutrino freeze-out.

(b) Whereas free neutrons decay quite rapidly, neutrons captured in a nucleus do not. Before neutrons decayed away, the formation of helium became possible. This allowed neutrons to survive.