Practice session: General Relativity

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.

1. **A torus in two-dimensional Euclidean space**

Consider the metric

$$ds^2 = (b + a\sin\phi)^2 d\theta^2 + a^2 d\phi^2 \tag{1}$$

Which describes a torus in 2D Euclidean space in the spherical coordinate system (θ, ϕ), where a and b are the torus radius and the radius of its section respectively.

(a) Derive the equations of motion for this metric for coordinate θ, using the Lagrangian.

(b) Read of the non-zero Christoffel symbols from your equations of motion derived in (a), using the formula

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{2}$$

(c) Also derive/compute all (consider both θ and ϕ) the non-zero Christoffel symbols 'by hand' using the formula

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right). \tag{3}$$

Solution:

(a) First the Lagrangian needs to be written down

$$\mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \sqrt{-\left[(b + a\sin\phi)^2\left(\frac{d\theta}{d\sigma}\right)^2 + a^2\left(\frac{d\phi}{d\sigma}\right)^2\right]}. \tag{4}$$

Subsequently, the equations of motion in θ can be derived,

$$\frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{1}{2\mathcal{L}}\frac{\partial}{\partial\dot{\theta}}\left[-\left[(b + a\sin\phi)^2\left(\frac{d\theta}{d\sigma}\right)^2 + a^2\left(\frac{d\phi}{d\sigma}\right)^2\right]\right] \tag{6}$$

$$= -\frac{1}{2\mathcal{L}}2(b + a\sin\phi)^2\frac{d\theta}{d\sigma} \tag{7}$$

$$= -(b + a\sin\phi)^2\frac{d\theta}{d\tau} \tag{8}$$

$$\tag{9}$$

Now the equation of motion looks as follows,

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{10}$$

$$\frac{d}{d\tau}\left(-(b + a\sin\phi)^2\frac{d\theta}{d\tau}\right) = 0 \tag{11}$$

$$-(b + a\sin\phi)^2\ddot{\theta} - (b + a\sin\phi)\dot{\theta}\cdot 2a\cos\phi\,\dot{\phi} = 0 \tag{12}$$

$$\ddot{\theta} + \frac{2a\cos\phi}{b + a\sin\phi}\dot{\phi}\dot{\theta} = 0. \tag{13}$$

(b) Looking at the equation of motion in (a), the non-zero Christoffel symbols are,

$$\Gamma^{\theta}_{\theta\phi} = \frac{1}{2} \cdot \frac{2a\cos\phi}{b + a\sin\phi} = \frac{a\cos\phi}{b + a\sin\phi}. \tag{14}$$

(c) Let's first consider the metric and its inverse. The metric is symmetric and can therefore trivially be inverted as follows

$$g_{\mu\nu} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} (b + a\sin\phi)^2 & 0 \\ 0 & a^2 \end{pmatrix} \Longrightarrow g^{\mu\nu} = \begin{pmatrix} (b + a\sin\phi)^{-2} & 0 \\ 0 & a^{-2} \end{pmatrix} \tag{15}$$

The derivatives of the metric w.r.t. the coordinates are then,

$$\frac{\partial g_{\mu\nu}}{\partial\theta} = 0 \tag{16}$$

$$\frac{\partial g_{\mu\nu}}{\partial\phi} = \begin{pmatrix} 2a(b + a\sin\phi)\cos\phi & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

$$\tag{18}$$

Noting that we only have two coordinates and the metric is diagonal, we get the following non-zero Christoffel symbols from the definition,

$$\Gamma^{\theta}_{\beta\gamma} = \Gamma^{\theta}_{\theta\phi} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial x^{\beta}} - \cancel{\frac{\partial g_{\beta\gamma}}{\partial x^{\theta}}} \right) \tag{19}$$

$$= \frac{1}{2} g^{\theta\theta} \cdot \frac{\partial g_{\theta\theta}}{\partial\phi} \tag{20}$$

$$= \frac{1}{2} \cdot \frac{1}{(b + a\sin\phi)^2} \cdot 2a(b + a\sin\phi)\cos\phi \tag{21}$$

$$= \frac{a\cos\phi}{b + a\sin\phi} \tag{22}$$

$$\Gamma^{\phi}_{\beta\gamma} = \Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\phi} \left(\cancel{\frac{\partial g_{\phi\beta}}{\partial x^{\gamma}}} + \cancel{\frac{\partial g_{\phi\gamma}}{\partial x^{\beta}}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\theta}} \right) \tag{23}$$

$$= -\frac{1}{2} g^{\phi\phi} \cdot \frac{\partial g_{\theta\theta}}{\partial\phi} \tag{24}$$

$$= -\frac{1}{2} \cdot \frac{1}{a^2} \cdot 2a(b + a\sin\phi)\cos\phi \tag{25}$$

$$= -\frac{(b + a\sin\phi)\cos\phi}{a} \tag{26}$$

$$\tag{27}$$

2. **Geometry of de Sitter space**

Consider the metric,

$$\mathrm{d}s^2 = (1 - r^2)\mathrm{d}t^2 - (1 - r^2)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{28}$$

which is a solution of Einstein equation with positive cosmological constant (de Sitter spacetime).

(a) Consider a massive particle undergoing inertial motion in this metric and derive its equations of motion.

(b) From this point on, we will be mostly concerned with motion in the (t, r) coordinates, so in what follows you can assume that the coordinates θ, ϕ always remain fixed. Show that a particle initially placed at the origin (i.e. $r|_{t=0} = 0$) with zero velocity (i.e. with $\frac{\mathrm{d}r}{\mathrm{d}t}|_{t=0} = 0$ will always stay there.

(c) Show that particles away from $r = 0$ feel a force towards larger values of, r, and will thus move towards the surface $r = 1$. On the surface, $r = 1$ the metric has a (coordinate) singularity. This surface is called the de Sitter horizon.

(d) Write an expression for the proper distance from $r = 0$ to the de Sitter horizon, and show that it is finite.

(e) Consider a light ray emitted from $r = 0$ towards the de Sitter horizon. Calculate its orbit and show that in the (t, r) coordinates the light ray never crosses the horizon.

(f) As in the case of the Schwarzschild black hole, this is somewhat misleading. Define the analogue of Eddington Finkelstein coordinates $\bar{t}$, r, in which the outgoing light rays (i.e. those moving towards large values of r) move along straight lines $\bar{t} - r = \mathit{constant}$.

(g) write the metric (28) in these new coordinates and show that it is smooth at $r = 1$ and can be extended past this surface to $r > 1$.

(h) Analyze the causal structure of the metric (28). This means, calculate the form of the in- and out-going lightcones, draw a spacetime diagram in the $\bar{t}$, r coordinates and plot qualitatively the form of the lightcones for ingoing (moving towards smaller r) and outgoing (moving towards larger r) lightrays.
From this diagram, argue that the surface $r = 1$ does indeed act like a horizon, that is, any object which starts in the region $r < 1$ and then crosses the surface $r = 1$ towards $r > 1$, will never be able to come back to the region $r < 1$. From the persepctive of an observer sitting at $r = 0$ this object is forever lost behind the de Sitter horizon.

Solution:

(a) In this case, the Lagrangian is given by:

$$\mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{29}$$

$$= \sqrt{-\left[(1-r^2)\left(\frac{dt}{d\sigma}\right)^2 - (1-r^2)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right)\right]}. \tag{30}$$

If we optimize the Lagrangian only in t we obtain:

$$\frac{\partial \mathcal{L}}{\partial t} = 0, \tag{31}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\partial}{\partial \dot{t}}\sqrt{-\left[(1-r^2)\left(\frac{dt}{d\sigma}\right)^2 - (1-r^2)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right)\right]} \tag{32}$$

$$= \frac{1}{2\mathcal{L}}\frac{\partial}{\partial \dot{t}}\left[-\left[(1-r^2)\left(\frac{dt}{d\sigma}\right)^2 - (1-r^2)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right)\right]\right] \tag{33}$$

$$= \frac{1}{2\mathcal{L}} \cdot -2(1-r^2)\frac{dt}{d\sigma} \tag{34}$$

$$= -\frac{d\sigma}{d\tau} \cdot (1-r^2)\frac{dt}{d\sigma} \tag{35}$$

$$= -(1-r^2)\frac{dt}{d\tau} = k, \tag{36}$$

where $\mathcal{L} = \frac{d\tau}{d\sigma}$ and $\dot{t} = \frac{dt}{d\sigma}$.

This means that we obtain for a massive particle:

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \vec{u}\cdot\vec{u} = -1 \tag{37}$$

$$-(1-r^2)\dot{t}^2 + \frac{\dot{r}^2}{1-r^2} = 1 \tag{38}$$

$$-(1-r^2)^2\dot{t}^2 + \dot{r}^2 = 1-r^2 \tag{39}$$

$$-k^2 + \dot{r}^2 = 1-r^2, \tag{40}$$

in which we were so generous to redefine the constant k. Also note for $\vec{u}\cdot\vec{u} = -1$, $ds^2 = -d\tau^2$.

(b) From the initial condition $r(0) = r_0 = 0$ and $\dot{r}(0) = 0$. We see that $k^2 = 1$ and the equation of motion for r becomes,

$$\dot{r} = r, \tag{41}$$

$$r(\tau) = r_0 e^{\tau} = 0. \tag{42}$$

Were we use the initial condition that $r_0 = 0$. This means that the particle at rest will stay at rest at $r = 0$.

(c) For $r_0 \neq 0$ (but small) we can use $r(\tau) = r_0 e^{\tau}$ to show that r grows exponential with τ. However in this approximation nothing special happens when we approach the horizon at $r = 1$.

(d) The proper distance from $r = 0$ to the horizon is given by:

$$L = \int_0^1 \frac{\mathrm{dr}}{\sqrt{1-r^2}} = \arcsin\Big|_0^1 = \frac{\pi}{2}. \tag{43}$$

(e) If we want to calculate something for a light-ray we don't use a proper-time constraints, but use the metric and the property that $\mathrm{ds}^2 = 0$ for light, radial motion so we have $\mathrm{d}\phi = \mathrm{d}\theta = 0$, so $\mathrm{ds}^2 = 0$ holds,

$$\mathrm{dt}^2 = \frac{\mathrm{dr}^2}{(1-r^2)^2}. \tag{44}$$

Integrate:

$$t + C = \int \frac{\mathrm{dr}}{1-r^2}, \tag{45}$$

$$= \int \frac{\mathrm{dr}}{(1-r)(1+r)}, \tag{46}$$

$$= \int \frac{\mathrm{dr}}{2}\left(\frac{1}{1+r} + \frac{1}{1-r}\right), \tag{47}$$

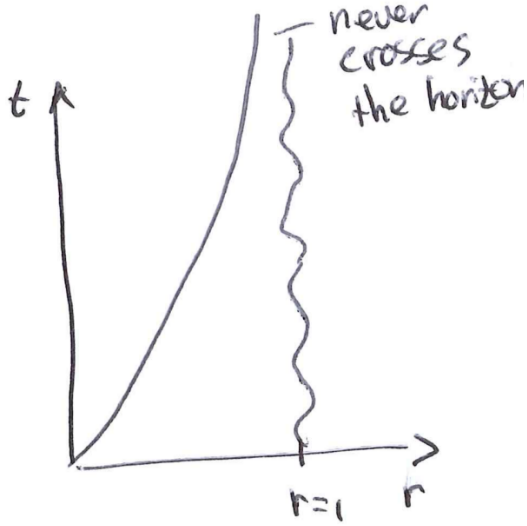
$$= \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right). \tag{48}$$

For a light ray emitted at $t = 0$, $r = 0$, the integration constant disappears. For small r we find:

$$\frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) \approx r. \tag{49}$$

And hence $t \approx r$.
As r approaches the horizon, $r \to 1$, and the denominator in the logarithm blows up,

so $t \to \infty$ as $r \to 1$. See the image which is shown below for a complete overview:



(f) Now we look for new coordinates, especially a new time coordinate $\bar{t}$ such that the equation of motion for an outgoing light ray is $\bar{t} - r = \text{const.}$. We modify the expression of (e):

$$t - \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) = \text{const.}, \tag{50}$$

$$= t - \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) + r - r. \tag{51}$$

So this means that:

$$\bar{t} = t + r - \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right). \tag{52}$$

and

$$r = r \tag{53}$$

(g) To write the metric in these coordinates we need,

$$\frac{\mathrm{dt}}{\mathrm{d}\tau} = \frac{\mathrm{d\bar{t}}}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-r + \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)\right), \tag{54}$$

$$= \frac{\mathrm{d\bar{t}}}{\mathrm{d}\tau} - \frac{\mathrm{dr}}{\mathrm{d}\tau} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\ln(1+r) - \ln(1-r)\right), \tag{55}$$

$$= \frac{\mathrm{d\bar{t}}}{\mathrm{d}\tau} - \frac{\mathrm{dr}}{\mathrm{d}\tau} + \frac{1}{2}\left(\frac{1}{1+r}\frac{\mathrm{dr}}{\mathrm{d}\tau} + \frac{1}{1-r}\frac{\mathrm{dr}}{\mathrm{d}\tau}\right), \tag{56}$$

$$= \frac{\mathrm{d\bar{t}}}{\mathrm{d}\tau} - \frac{\mathrm{dr}}{\mathrm{d}\tau}\left(\frac{1}{1-r^2} - 1\right), \tag{57}$$

$$= \frac{\mathrm{d\bar{t}}}{\mathrm{d}\tau} + \frac{r^2}{1-r^2}\frac{\mathrm{dr}}{\mathrm{d}\tau}. \tag{58}$$

So this means that:

$$\mathrm{dt} = \mathrm{d\bar{t}} + \frac{\mathrm{r}^2}{1-\mathrm{r}^2}\mathrm{dr}. \tag{59}$$

For this exercise we ignore the θ and ϕ part, because it remains the same. If we rewrite the metric we obtain:

$$\begin{aligned} ds^2 &= (1-r^2)dt^2 - (1-r^2)^{-1}dr^2, && (60)\\ &= (1-r^2)\left(d\bar{t} + \frac{r^2}{1-r^2}dr\right)^2 - (1-r^2)^{-1}dr^2, && (61)\\ &= (1-r^2)d\bar{t}^2 + 2r^2 d\bar{t}dr + \frac{r^4}{1-r^2}dr^2 - \frac{1}{1-r^2}dr^2, && (62)\\ &= (1-r^2)d\bar{t}^2 + 2r^2 d\bar{t}dr + \frac{r^4-1}{1-r^2}dr^2, && (63)\\ &= (1-r^2)d\bar{t}^2 + 2r^2 d\bar{t}dr - (1+r^2)dr^2. && (64)\end{aligned}$$

Using the fact that $-(1-r^2)(1+r^2) = r^4 - 1$.
This result means the metric is smooth everywhere and can be extended beyond $r = 1$.

(h) We are now going to investigate the propagation of all light rays moving in radial direction. So we solve:

$$ds^2 = (1-r^2)d\bar{t}^2 + 2r^2 d\bar{t}dr - (1+r^2)dr^2. \tag{65}$$

Deviding by dr^2 yields a quadratic equation for $\frac{d\bar{t}}{dr} \equiv \lambda$:

$$(1-r^2)\lambda^2 + 2r^2\lambda - (1-r^2) = 0. \tag{66}$$

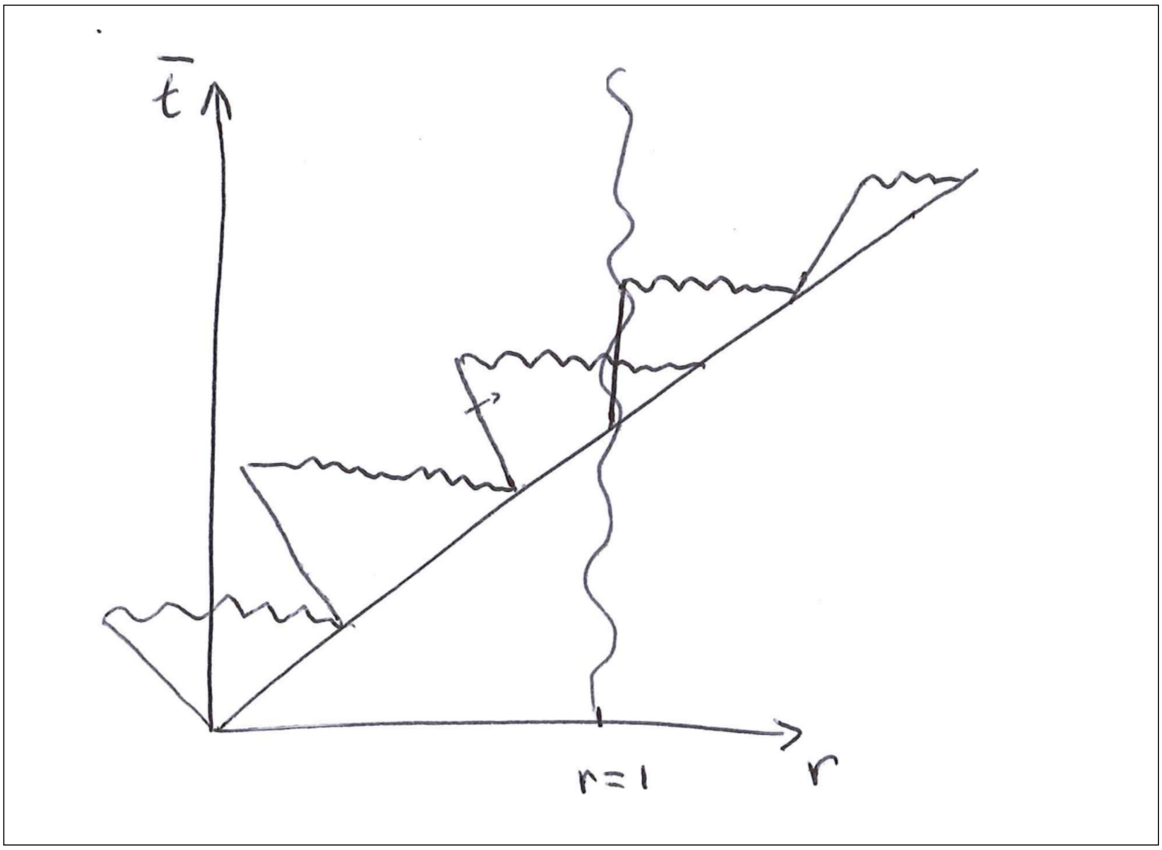
So there are two solutions:

$$\begin{aligned} \lambda &= \frac{-2r^2 \pm \sqrt{4r^4 - 4\cdot(1-r^2)\cdot -(1+r^2)}}{2(1-r^2)}, && (67)\\ &= \frac{-2r^2 \pm \sqrt{4}}{2(1-r^2)}, && (68)\\ &= \frac{-r^2 \pm 1}{1-r^2}. && (69)\end{aligned}$$

This means the solutions are given by:

$$\begin{aligned} \lambda_+ &= 1, && (70)\\ \lambda_- &= -\frac{1+r^2}{1-r^2}. && (71)\end{aligned}$$

The result is that the outgoing angle is precisely 45 degrees. For the negative solution we have a different result that implies that for small r we have an angle of minus 45 degrees. However we not that as $r \to 1$, λ_- diverges and for $r > 1$ it even becomes positive, ingoing and outgoing light rays start to travel in the same direction. This means once a particle has crossed the horizon at $r = 1$ it has to stay within the light cone preventing it from returning to values $r < 1$.



3. **Expanding flat universe**

Consider the following metric describing an expanding flat Universe,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{72}$$

(a) Compute the Christoffel symbols in terms of $a(t)$.

(b) Derive the equations of motion for massive particles moving inertially in this spacetime

(c) Check that the orbits {x(t), y(t), z(t)} = constant, correspond to inertial motion.

From this point on, we focus on the specific case of an inflating Universe where,

$$a(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{73}$$

(d) A light ray is emitted from the point {t, x, y, z} = {t_0 , 0, 0, 0} towards positive values of x. What is the orbit that the light ray will follow?

(e) Find the maximum value of the coordinate x that the light ray described above can reach.

(f) Using the metric compute the physical spacelike distance along the slice $t = t_0$ corresponding to the coordinate distance in x that you found in e).

Solution:

(a) So we consider the following metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{74}$$

This gives a Lagrangian given by:

$$\mathcal{L} = \sqrt{-\left[\left(\frac{dt}{d\sigma}\right)^2 - a(t)^2\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]}. \tag{75}$$

This mean we obtain for t:

$$\frac{\partial\mathcal{L}}{\partial t} = \frac{1}{2\mathcal{L}}\frac{\partial}{\partial t}\left[-\left[\left(\frac{dt}{d\sigma}\right)^2 - a(t)^2\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]\right] \tag{76}$$

$$= \frac{1}{2\mathcal{L}}2a(t)\frac{\partial a}{\partial t}\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right) \tag{77}$$

$$= a(t)\frac{\partial a}{\partial t}\left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right), \tag{78}$$

$$\frac{\partial\mathcal{L}}{\partial \dot{t}} = \frac{1}{2\mathcal{L}}\frac{\partial}{\partial \dot{t}}\left[-\left[\left(\frac{dt}{d\sigma}\right)^2 - a(t)^2\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]\right] \tag{79}$$

$$= -\frac{1}{2\mathcal{L}}2\frac{dt}{d\sigma} \tag{80}$$

$$= -\frac{dt}{d\tau}, \tag{81}$$

where $\mathcal{L} = \frac{d\tau}{d\sigma}$.
This leads to the following Lagrange equation and means that the Christoffel symbols become:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) - \frac{\partial\mathcal{L}}{\partial t} = 0 \tag{82}$$

$$\frac{\partial}{\partial\tau}\left(-\frac{dt}{d\tau}\right) - a(t)\frac{\partial a(t)}{\partial t}\left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right) = 0 \tag{83}$$

$$\frac{d^2t}{d\tau^2} + a\frac{\partial a}{\partial t}\left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right) = 0, \tag{84}$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\frac{\partial a}{\partial t}. \tag{85}$$

Now let's consider x or a cyclic perturbation of x, y, z:

$$\frac{\partial\mathcal{L}}{\partial x} = 0, \tag{86}$$

$$\frac{\partial\mathcal{L}}{\partial \dot{x}} = \frac{1}{2\mathcal{L}}\frac{\partial}{\partial \dot{x}}\left[-\left[\left(\frac{dt}{d\sigma}\right)^2 - a(t)^2\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]\right] \tag{87}$$

$$= \frac{1}{2\mathcal{L}}a(t)^2 2\frac{dx}{d\sigma} \tag{88}$$

$$= a(t)^2\frac{dx}{d\sigma}. \tag{89}$$

The Lagrange equation can then be solved to obtain the Christoffel symbols,

$$\frac{d}{d\tau}\left(\frac{d\mathcal{L}}{d\dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{90}$$

$$\frac{d}{d\tau}\left(a(t)^2 \frac{dx}{d\tau}\right) = 0 \tag{91}$$

$$2a(t)\frac{da(t)}{dt}\frac{dt}{d\tau}\frac{dx}{d\tau} + a(t)^2 \frac{d^2x}{d\tau^2} = 0 \tag{92}$$

$$\frac{d^2x}{d\tau^2} + \frac{2}{a(t)}\frac{\partial a}{\partial t}\frac{dt}{d\tau}\frac{dx}{d\tau}. \tag{93}$$

This implies that:

$$\Gamma^x_{xt} = \Gamma^y_{yt} = \Gamma^z_{zt} = \frac{1}{a}\frac{\partial a}{\partial t} \tag{94}$$

Mind the factor 2, due to symmetry.

(b) For a massive particle we know that:

$$1 = \dot{t}^2 - a(t)^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{95}$$

From the previous subproblem (a) we know that:

$$2a^2\dot{x} = k. \tag{96}$$

Because of symmetry this means that we have:

$$\dot{x} = \frac{k}{2a^2}, \tag{97}$$

$$\dot{y} = \frac{l}{2a^2}, \tag{98}$$

$$\dot{z} = \frac{m}{2a^2}, \tag{99}$$

This means that the equation for a massive particle becomes:

$$1 = \dot{t}^2 - \frac{1}{4a^2}\left(k^2 + l^2 + m^2\right). \tag{100}$$

This can be rewritten as:

$$\dot{t}^2 = 1 + \frac{1}{4a^2}\left(k^2 + l^2 + m^2\right). \tag{101}$$

(c) If we have {x(t), y(t), z(t) } = constant, this means that:

$$\dot{x} = \frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial t}\frac{\partial t}{\partial \tau} = 0 \cdot \frac{\partial t}{\partial \tau} = 0. \tag{102}$$

The same is valid for the cyclic permutable coordinates. so $\dot{x} = \dot{y} = \dot{z} = 0$. This gives that the equations of motion give:

$$\ddot{t} = \ddot{x} = \ddot{y} = \ddot{z} = 0. \tag{103}$$

So there is inertial motion.

(d) For now we assume:

$$a(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{104}$$

A light ray so that means that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, this gives:

$$0 = \mathrm{dt}^2 - \exp\left(\sqrt{\frac{\Lambda}{3}}\mathrm{t}\right)\mathrm{dx}^2, \tag{105}$$

$$\mathrm{dt}^2 = \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)\mathrm{dx}^2, \tag{106}$$

$$\mathrm{dx} = \exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t\right)\mathrm{dt}, \tag{107}$$

$$x(t) = -\frac{\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t\right)}{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}} + C. \tag{108}$$

Now we want to do set boundary conditions: $t = t_0$, $x(t_0) = 0$, this means the constant becomes:

$$C = \frac{\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right)}{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}}. \tag{109}$$

This means the equation for $x(t)$ becomes:

$$x(t) = \left(\frac{\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right)}{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}} - \frac{\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t\right)}{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}}\right), \tag{110}$$

$$= 2\sqrt{\frac{3}{\Lambda}}\left(\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right) - \exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t\right)\right) \tag{111}$$

(e) For this we take $t \to \infty$, which gives:

$$\lim_{t\to\infty} x(t) = 2\sqrt{\frac{3}{\Lambda}}\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right) \tag{112}$$

(f) We take a slice in time, $t = t_0$, this means that $\mathrm{dt} = 0$, and the metric becomes:

$$\mathrm{ds}^2 = a^2\mathrm{dx}^2, \tag{113}$$

$$\mathrm{ds} = a\mathrm{dx}, \tag{114}$$

$$s = a(t_0)x_{\max}(t_0), \tag{115}$$

$$= \exp\left(\sqrt{\frac{\Lambda}{3}}t_0\right)\cdot 2\sqrt{\frac{3}{\Lambda}}\exp\left(-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right), \tag{116}$$

$$= 2\sqrt{\frac{3}{\Lambda}}\exp\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t_0\right). \tag{117}$$

4. **General to Special Relativity**

Assuming a general coordinate transformation, the components of vector fields **a** and **b** transform as follows:

$$a'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} a^{\beta}, \qquad b'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} b_{\beta}, \tag{118}$$

(a) Show that the components of a metric tensor transform under general coordinate transformations as:

$$g'_{\alpha\beta}(x) = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}(x) \tag{119}$$

(b) Show that the scalar product between two vector fields:

$$\boldsymbol{v}(x) \cdot \boldsymbol{w}(x) = g_{\alpha\beta}(x) v^{\alpha}(x) w^{\beta}(x) \tag{120}$$

is invariant under the general coordinate transformations.

(c) Show that under a general coordinate transformation ($x'^{\alpha} = (\Lambda^{-1})^{\alpha}{}_{\beta} x^{\beta}$), where Λ^{-1} is a constant invertable matrix, the components of vector field **b** transform as:

$$b'_{\alpha} = \Lambda^{\beta}_{\alpha} b_{\beta} \tag{121}$$

(d) Now consider a local inertial frame at a Point P where $g_{\alpha\beta} = \eta_{\alpha\beta}$ and the first derivative of $g_{\alpha\beta}$ vanishes. Using eq. (119), show that the components of the Minkowski spacetime metric $\eta_{\alpha\beta}$ are invariant under general coordinate transformations (as defined in the previous subquestion). In other words, show that the conditions for which $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$ are given by:

$$(\Lambda^{T})^{\alpha}_{\gamma} (\Lambda)^{\gamma}_{\beta} = \mathbb{1}^{\alpha}_{\beta} \quad \text{where} \quad (\Lambda^{T})^{\alpha}_{\beta} \equiv \eta^{\alpha\gamma} \eta_{\beta\delta} \Lambda^{\delta}_{\gamma} \tag{122}$$

Solution:

(a) The metric tensor components can be written as the product of 2 vectors:

$$g'_{\alpha\beta} = a'_{\alpha} b'_{\beta} \tag{123}$$

Now substituting the expression for a general coordinate transformation gives:

$$g'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} a_{\gamma} \frac{\partial x^{\delta}}{\partial x'^{\beta}} b_{\delta} \tag{124}$$

$$= \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} a_{\gamma} b_{\delta} \tag{125}$$

$$= \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta} \tag{126}$$

(b) Take the scalar product definition and fill in the transformation equations previously specified in the question.

$$\boldsymbol{v'}(x) \cdot \boldsymbol{w'}(x) = g'_{\alpha\beta}(x) v'^{\alpha}(x) w'^{\beta}(x) \tag{127}$$

$$= \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}(x) \frac{\partial x'^{\alpha}}{\partial x^{\nu}} a^{\nu} \frac{\partial x'^{\beta}}{\partial x'^{\mu}} b^{\mu} \tag{128}$$

$$= \delta^{\gamma}_{\nu} \delta^{\delta}_{\mu} g_{\gamma\delta} a^{\nu} b^{\mu} \tag{129}$$

$$= g_{\gamma\delta} a^{\gamma} b^{\delta} \tag{130}$$

$$= \boldsymbol{a}(x) \cdot \boldsymbol{b}(x) \tag{131}$$

So yes, the scalar product between two vector fields in invariant under the general coordinate transformations.

(c) Start with the general coordinate transformation:

$$x'^{\alpha} = (\Lambda^{-1})^{\alpha}{}_{\beta} x^{\beta} \tag{132}$$

Then multiply both sides by $\Lambda^{\delta}_{\alpha}$.

$$\begin{aligned} \Lambda^{\delta}_{\alpha} x'^{\alpha} &= \Lambda^{\delta}_{\alpha} (\Lambda^{-1})^{\alpha}{}_{\beta} x^{\beta} && (133) \\ &= \delta^{\delta}_{\beta} x^{\beta} && (134) \\ &= x^{\delta} && (135) \end{aligned}$$

We can also write this as follows (to be consistent with the notation of the question.

$$x^{\beta} = \Lambda^{\beta}_{\alpha} x'^{\alpha} \quad \Rightarrow \quad \frac{\partial x^{\beta}}{\partial x'^{\alpha}} = \Lambda^{\beta}_{\alpha} \tag{136}$$

We can substitute this in the transformation of a vector field to get the final solution:

$$\begin{aligned} b'_{\alpha} &= \frac{\partial x^{\beta}}{\partial x'^{\alpha}} b_{\beta} && (137) \\ &= \Lambda^{\beta}_{\alpha} b_{\beta} && (138) \end{aligned}$$

(d) We can rewrite eq. (119) using eq. (121), and set $g_{\alpha\beta} = \eta_{\alpha\beta}$, which gives:

$$\eta'_{\alpha\beta} = \Lambda^{\gamma}_{\alpha} \Lambda^{\delta}_{\beta} \eta_{\gamma\delta} \tag{139}$$

Then multiply this expression by $\eta^{\epsilon\alpha}$

$$\delta^{\epsilon}_{\beta} = \eta^{\epsilon\alpha} \eta_{\gamma\delta} \Lambda^{\gamma}_{\alpha} \Lambda^{\delta}_{\beta} \tag{140}$$

Additionally, we know that:

$$\begin{aligned} \eta_{\alpha\beta} = \eta'_{\alpha\beta} &= a'_{\alpha} b'_{\beta} && (141) \\ &= \Lambda^{\gamma}_{\alpha} \Lambda^{\delta}_{\beta} a_{\gamma} b_{\delta} && (142) \\ &= \Lambda^{\gamma}_{\alpha} \Lambda^{\delta}_{\beta} \eta_{\gamma\delta} && (143) \end{aligned}$$

or equivalently, $(\Lambda^T)^{\epsilon}_{\delta} = \eta^{\epsilon\alpha} \eta_{\gamma\delta} \Lambda^{\gamma}_{\alpha} \Lambda^{\delta}_{\beta}$
Substituting this in eq. (140) gives

$$\delta^{\epsilon}_{\beta} = (\Lambda^T)^{\epsilon}{}_{\delta} \Lambda^{\delta}_{\beta} = \mathbb{1}^{\epsilon}_{\beta} \tag{144}$$

QED

5. **Einstein & Riemann**

(a) Show that Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{145}$$

can also be written as,

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + \frac{T}{2-D} g_{\mu\nu} \right), \tag{146}$$

in which D is the dimension of space time.

(b) How many independent components does the Riemann tensor have in 2 dimensions?[1]

Solution:

(a)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{147}$$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = 8\pi G T_{\mu\nu}g^{\mu\nu}, \tag{148}$$

$$R - \frac{D}{2}R = 8\pi G T, \tag{149}$$

$$R\left(1 - \frac{D}{2}\right) = 8\pi G T, \tag{150}$$

$$R = \frac{8\pi G T}{1 - \frac{D}{2}}, \tag{151}$$

where $D = g^{\mu\nu}g_{\mu\nu}$.
Now substituting back into Einstein's equations,

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R, \tag{152}$$

$$= 8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\frac{8\pi G T}{1 - \frac{D}{2}}, \tag{153}$$

$$= 8\pi G\left(T_{\mu\nu} + \frac{T}{2 - D}g_{\mu\nu}\right). \tag{154}$$

(b) This exercise is a common exercise which need to be fluent by the student. Best to approach this problem is by writing all possibilities as counting in binary:

R_{1111}	R_{1211}	R_{2111}	R_{2211}
R_{1112}	R_{1212}	R_{2112}	R_{2212}
R_{1121}	R_{1221}	R_{2121}	R_{2221}
R_{1122}	R_{1222}	R_{2122}	R_{2222}

We know that the metric is antisymmetric in the first two indices and the last two indices, this means that the following become zero:

$\cancel{R_{1111}}$	$\cancel{R_{1211}}$	$\cancel{R_{2111}}$	$\cancel{R_{2211}}$
$\cancel{R_{1112}}$	R_{1212}	R_{2112}	$\cancel{R_{2212}}$
$\cancel{R_{1121}}$	R_{1221}	R_{2121}	$\cancel{R_{2221}}$
$\cancel{R_{1122}}$	$\cancel{R_{1222}}$	$\cancel{R_{2122}}$	$\cancel{R_{2222}}$

These four remaining Riemann tensor components can therefore be related using the anti-symmetric properties as:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} \tag{155}$$

So this mean there is only 1 independent component. The number of independent components can also be found using:

$$\mathrm{IC} = \frac{1}{12}N^2(N^2 - 1) \tag{156}$$

[1] Show explicitly that there are that amount of independent components.