General Relativity 2025-2026: Assignment 2

posted at October 7, to be submitted on October 14

N.B. you can obtain a maximum of **(140pt)**

Problem 1: Schwarzschild Black Hole (30 pt)

Consider the trajectory of an astronaut in the presence of a black hole described by the Schwarzschild geometry (with $G = c = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{1}$$

described by the geodesic $x^\alpha(\tau) = \big(t(\tau), r(\tau), \theta(\tau), \phi(\tau)\big)$. Without loss of generality you may assume that $\theta(\tau) = \pi/2$.

a. Show that the quantity

$$\ell = r^2 \frac{d\phi}{d\tau} \tag{2}$$

is conserved along the geodesic. What is the physical interpretation of ℓ? (10 pt)

b. One of the geodesic equations of the astronaut is given by

$$\frac{M}{r^2}\Big(\frac{dt}{d\tau}\Big)^2 - r\Big(\frac{d\phi}{d\tau}\Big)^2 = 0\,. \tag{3}$$

The astronaut finds himself in free fall in a circular orbit of radius $R = 4M$. Calculate the coordinate time Δt in terms of M that the astronaut needs for one revolution. (10 pt)

c. Calculate also the proper time $\Delta\tau$ in terms of M that the astronaut needs for the same revolution. (10 pt)

Problem 2: Cosmology (40 pt)

Consider the Friedman equation

$$\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = 1\,, \tag{4}$$

where $a(t)$ is the scale factor and $\rho(t)$ is the total energy density.

For a universe with pressureless matter the total energy density $\rho(t)$ is given by

$$\rho(t) = \rho(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3, \tag{5}$$

where t_0 is the present-day time and

$$\rho(t_0) = \frac{3}{8\pi} H_0^2\,\Omega\,, \qquad H_0 = \frac{\dot{a}(t_0)}{a(t_0)}\,. \tag{6}$$

a. Using eqs. (5) and (6), show that the Friedman equation, taken at $t = t_0$, can be used to solve for $a_0 = a(t = t_0)$ in terms of H_0 and Ω as follows: (10 pt)

$$a_0 = (H_0)^{-1} (1 - \Omega)^{-1/2} . \tag{7}$$

b. Use the result (7) to show that the Friedman equation, for general t, can be written as (10 pt)

$$\dot{a}^2 - 1 = \frac{A^2}{a} \qquad \text{with} \qquad A^2 = \frac{\Omega}{H_0(1-\Omega)^{3/2}} . \tag{8}$$

c. Show that equation (8) is solved by (10 pt)

$$a(\eta) = \frac{1}{2} A^2 (\cosh \eta - 1) , \qquad\qquad t(\eta) = \frac{1}{2} A^2 (\sinh \eta - \eta) \tag{9}$$

in terms of a parameter η.

d. Show using a schematic diagram of a versus t whether the solution (9) describes an open or closed universe and give the value of the time t for which a possible Big Bang and/or Big Crunch occurs. (10 pt)

Problem 3: The Hyperbolic Plane (70pt)

The hyperbolic plane defined by the metric

$$dS^2 = y^{-2}(dx^2 + dy^2) , \qquad y > 0 \tag{10}$$

is a classic example of a two-dimensional surface with hyperbolic geometry.

a. Using a variational principle, derive the Lagrangian for a curve $x^\alpha(\sigma) = \big(x(\sigma), y(\sigma)\big)$ in terms $x^\alpha(\sigma)$ and $dx^\alpha/d\sigma$. (10pt)

b. Using the Euler-Lagrange equations that correspond to this Lagrangian, derive the following geodesic equations for x and y: (20pt)

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 , \qquad \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0 , \tag{11}$$

where the dot refers to a differentiation with respect to S, i.e. $\dot{x} \equiv dx/dS$. Note that $\dot{x}$ is not the same as $dx/d\sigma$.

c. Read off from these equations the expressions of the Christoffel symbols. (10pt)

d. Show a symmetry of the line element (10) and use this symmetry to derive a conserved quantity. (10pt)

e. Use the first geodesic equation in (11) together with the fact that $\vec{u} \cdot \vec{u} = 1$ to find a quadratic relation between x and y. Show that this quadratic relation represents a family of semi-circles in the upper half plane of arbitrary radius r

centered at arbitrary position $(x_0, 0)$ at the x-axis. Hint: Use your answer to question d. You will also need the following integral:

$$\int \frac{y\,dy}{\sqrt{r^2 - y^2}} = -\sqrt{r^2 - y^2} \tag{12}$$

for some constant r. (20pt)