General Relativity 2025-2026: Assignment 1

posted at September 16, to be submitted on September 23

N.B. you can obtain a maximum of **(200pt)**

Problem 1: Indices (30pt)

a. Indicate in each of the following expressions whether the index α is a free index or a dummy index. (15pt)

$$a^\alpha b_\alpha = 1\,, \qquad a^\alpha b^\beta c_\beta = d^\alpha\,, \qquad a^\alpha b^\beta = d^{\alpha\beta}\,. \tag{1}$$

b. Indicate in which of the following expressions the Einstein summation convention is used. (15pt)

$$a^\alpha b_\beta = c^\alpha{}_\beta\,, \qquad a^\alpha b_\alpha = 1\,, \qquad c^\alpha{}_\alpha = 1\,. \tag{2}$$

Problem 2: Scalar product (30pt)

a. Consider a spacetime with coordinates (t, x) and line element

$$ds^2 = -dt^2 + dtdx + dx^2\,. \tag{3}$$

Write down the metric tensor components $g_{\alpha\beta}$ corresponding to this line element. (10pt)

b. Consider in this spacetime a vector

$$V^\alpha = \begin{pmatrix} z \\ 1 \end{pmatrix} \tag{4}$$

with z a real number. Calculate the scalar product $\vec{V} \cdot \vec{V}$. (10pt)

c. For which values of z is the vector $\vec{V}$ spacelike, lightlike or timelike? (10pt)

Problem 3: Calculating Distances and Areas (20pt)

Consider a spacetime geometry with coordinates (t, r, θ, ϕ) and line element

$$ds^2 = -(1 - Ar^2)^2\, dt^2 + (1 - Ar^2)^2\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

for certain constant A.

a. Calculate the proper distance along a radial line at constant t from the centre $r = 0$ to a coordinate radius $r = R$. (10pt)

b. Calculate the area of a sphere of coordinate radius $r = R$. (10pt)

Problem 4: Local lightcones (60pt)

Consider a spacetime geometry with coordinates (t, x, y, z) and line element

$$ds^2 = -dt^2 + e^{2Ht}\big(dx^2 + dy^2 + dz^2\big) \tag{6}$$

for constant H.

a. Determine the non-zero metric components and calculate its determinant. (10pt)

b. Calculate the light ray trajectories in this geometry and indicate them in a spacetime diagram. *Hint:* Assume fixed (y, z) coordinates and restrict to a two-dimensional (t, x). (20pt)

c. Show, by indicating a few local lightcones in the (t, x) spacetime diagram, that these lightcones become more narrow with increasing time. (10pt)

d. Find a coordinate transformation $t = t(\eta)$ such that

$$ds^2 = \Omega^2(\eta)\Big(- d\eta^2 + dx^2 + dy^2 + dz^2\Big) \tag{7}$$

for some $\Omega(\eta)$. Also write down $\Omega(\eta)$ explicitly. (20pt)

Problem 5: Christoffel Symbols (60pt)

Consider a three-dimensional spacetime with coordinates $x^\alpha = (t, r, \phi)$ and line element:

$$ds^2 \; = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \,. \tag{8}$$

a. Show that the Lagrangian for the variational principle for geodesics $x^\alpha(\sigma)$ in this spacetime is given by (10pt)

$$L(\dot t, \dot r, \dot\phi, r) = \left[\left(1 - \frac{2M}{r}\right)\dot t^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot r^2 - r^2\dot\phi^2\right]^{1/2} \tag{9}$$

with $\dot x^\alpha = dx^\alpha/d\sigma$.

b. Vary the Lagrangian (9) with respect to the ϕ coordinate and show that the corresponding Euler-Lagrange equation of motion is given by (20pt)

$$\frac{d}{d\tau}\left[r^2\frac{d\phi}{d\tau}\right] = 0\,. \tag{10}$$

c. Using this equation of motion read off the expressions for the Christoffel symbols $\Gamma^\phi_{\alpha\beta}$ for all α, β by comparing with the general expression of the geodesic equation (10pt)

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0\,. \tag{11}$$

d. Calculate the expression for the Christoffel symbols $\Gamma^{\phi}_{\alpha\beta}$ also 'by hand' using the formula

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right) \tag{12}$$

and compare your result with the expression calculated in c. (20pt)