PROBLEM 1

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a) $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$

is the Schwarzschild metric in geometrical units.

The metric is independent of t, the Killing vector is $\xi^\alpha = (1,0,0,0)$.

there is also a manifest rotational simmetry, as the metric is independent of ϕ, with Killing vector $\eta^\alpha = (0,0,0,1)$ and

$$\frac{\partial g_{\alpha\beta}}{\partial\phi} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0$$

Euler - Lagrange eq:

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{d\phi}{d\sigma}\right)}\right) + \frac{\partial L}{\partial\phi} = 0 \longrightarrow -\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{d\phi}{\partial\sigma}\right)}\right) = 0$$

and

$$\frac{\partial L}{\partial\left(\frac{d\phi}{d\sigma}\right)} = \text{const} \qquad L = \left[-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right]^{\frac{1}{2}}$$

$$\frac{\partial L}{\partial\left(\frac{d\phi}{d\sigma}\right)} = \frac{1}{2}\left(L^2\right)^{-\frac{1}{2}}\left(-g_{3\beta}\frac{dx^\beta}{d\sigma}\right) = -\frac{1}{2L}\, g_{3\beta}\frac{dx^\beta}{d\tau}\overbrace{\frac{d\tau}{d\sigma}}^{L} = \text{const}$$

$\longrightarrow g_{\alpha\beta}\eta^\alpha u^\beta = \underset{\wedge}{\eta}\cdot\underset{\wedge}{u} = \text{const} \longrightarrow \underset{\wedge}{\eta}\cdot\underset{\wedge}{u}$ is conserved

$$\eta\cdot u = g_{\phi\phi}\frac{d\phi}{d\tau} = r^2\sin^2\theta\frac{d\phi}{d\tau} = r^2\frac{d\phi}{d\tau} \qquad \text{assuming } \theta(\tau) = \frac{\pi}{2}$$

the quantity $\ell = r^2\frac{d\phi}{d\tau}$ is conserved along the geodesic.

it is the angular momentum per unit mass.

b) $\frac{M}{r^2}\left(\frac{dt}{d\tau}\right)^2 - r\left(\frac{d\phi}{d\tau}\right)^2 = 0$ is the equation of the geodesic

circular orbit of radius $R = 4M$

at $r = R$: $M\left(\frac{dt}{d\tau}\right)^2 - R^3\left(\frac{d\phi}{d\tau}\right)^2$ $\quad R = 4M$

$\left(\frac{dt^2}{d\tau}\right)^2 - 64 M^2 \left(\frac{d\phi}{d\tau}\right)^2 = 0 \quad \rightarrow \quad \frac{dt}{d\tau} = 8M \frac{d\phi}{d\tau}$

$\Delta t = 8M \oint d\phi = 8M \int_0^{2\pi} d\phi = 16\pi M$ $\quad \Delta t = 16\pi M$

c) $u \cdot u = -1$ and $\theta = \frac{\pi}{2}$

$g_{\alpha\beta} u^\alpha u^\beta = -1$

$g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = -1$

$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = 1$

on the geodesic $r = 4M = R$, $dr = 0$

$-\left(1 - \frac{2M}{4M}\right)\left(\frac{dt}{d\tau}\right)^2 + 16 M^2 \left(\frac{d\phi}{d\tau}\right)^2 = -1$

from point b) : $\left(\frac{dt}{d\tau}\right)^2 = 64 M^2 \left(\frac{d\phi}{d\tau}\right)^2$

$-\frac{1}{2} \cdot 64 M^2 \left(\frac{d\phi}{d\tau}\right)^2 + 16 M^2 \left(\frac{d\phi}{d\tau}\right)^2 = -1$

$16 M^2 \left(\frac{d\phi}{d\tau}\right)^2 = 1 \longrightarrow \frac{d\phi}{d\tau} = \frac{1}{4M}$

$\Delta\tau = \oint d\tau = \int_0^{2\pi} \frac{d\tau}{d\phi} d\phi = 4M \int_0^{2\pi} d\phi = 8\pi M$ $\quad \Delta\tau = 8\pi M$

PROBLEM 2

$$\boxed{\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = 1}$$

is the Friedman equation, where

$\rho(t) = \rho(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$ is the total energy density

$\rho(t_0) = \frac{3}{8\pi} H_0^2 \Omega$, $H_0 = \frac{\dot{a}(t_0)}{a(t_0)}$

here $k = -1$, $\Omega < 1$ that is negative curvature
(open universe)

a) $\rho(t) = \frac{3}{8\pi} H_0^2 \Omega \left(\frac{a_0}{a}\right)^3$ where $a_0 = a(t_0)$

substituting in the Friedman equation:

$$\dot{a}^2 - \frac{8\pi}{3} \cdot \frac{3}{8\pi} H_0^2 \Omega a^2 \frac{a_0^3}{a^3} = 1$$

$$\dot{a}^2 - H_0^2 \Omega \frac{a_0^3}{a} = 1$$

let $t = t_0$ and multiply with $\frac{1}{a_0^2}$

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 - H_0^2 \Omega \frac{a_0^3}{a_0^3} = \frac{1}{a_0^2}$$

$$H_0^2 - H_0^2 \Omega = \frac{1}{a_0^2} \rightarrow a_0^2 H_0^2 (1-\Omega) = 1 \quad \text{with } \Omega < 1$$

so we get:

$$a_0 = H_0^{-1} (1-\Omega)^{-\frac{1}{2}}$$

b) let $A^2 = \frac{\Omega}{H_0 (1-\Omega)^{3/2}}$

from point a):

$$\left.\begin{aligned} a_0 H_0 &= (1-\Omega)^{-\frac{1}{2}} \\ \dot{a}^2 - a_0^3 \frac{H_0^3}{H_0} \frac{\Omega}{a} &= 1 \end{aligned}\right\} \rightarrow \dot{a}^2 - \frac{\Omega}{H_0 (1-\Omega)^{3/2}} \cdot \frac{1}{a} = 1$$

and

$$\dot{a}^2 - \frac{A^2}{a} = 1 \longrightarrow \boxed{\dot{a}^2 - 1 = \frac{A^2}{a}}$$

c) $\dot{a} - 1 = \frac{A^2}{a}$ is the Friedman equation

$$\begin{cases} a(\eta) = \frac{1}{2} A^2 (\cosh\eta - 1) \\ t(\eta) = \frac{1}{2} A^2 (\sinh\eta - \eta) \end{cases}$$

are the parametric equations for the negative curvature models $K = -1$, $\Omega < 1$

η is the conformal time: $dt = a(t)\, d\eta$

$$\dot{a} = \frac{da}{dt} = \frac{da/d\eta}{dt/d\eta} = \frac{\frac{1}{2} A^2 \sinh\eta}{\frac{1}{2} A^2 (\cosh\eta - 1)} = \frac{\sinh\eta}{\cosh\eta - 1}$$

(1) $$\dot{a}^2 - 1 = \frac{\sinh^2\eta}{(\cosh\eta - 1)^2} - 1 = \frac{\sinh^2\eta - (\cosh\eta - 1)^2}{(\cosh\eta - 1)^2} =$$

$$= \frac{\sinh^2\eta - \cosh^2\eta + 2\cosh\eta - 1}{(\cosh\eta - 1)^2} \qquad (\cosh^2\eta - \sinh^2\eta = 1)$$

$$= \frac{-1 + 2\cosh\eta - 1}{(\cosh\eta - 1)^2} = \frac{2(\cosh\eta - 1)}{(\cosh\eta - 1)^2} = \frac{2}{\cosh\eta - 1}$$

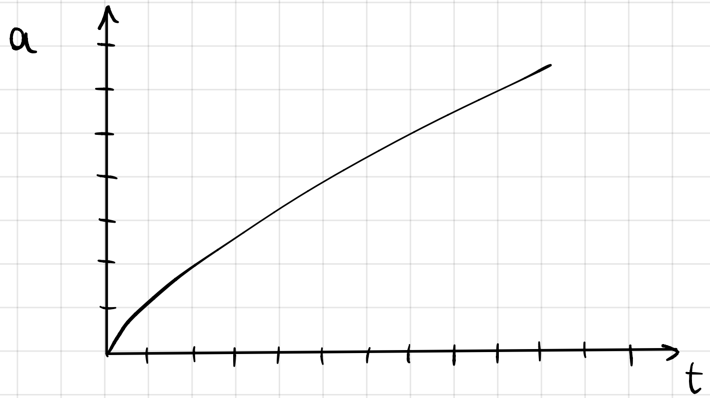
(2) $$\frac{A^2}{a} = \frac{A^2}{dt/d\eta} \qquad \frac{A^2}{\frac{1}{2} A^2 (\cosh\eta - 1)} = \frac{2}{\cosh\eta - 1}$$

(1) = (2) $\rightarrow$ the Friedman equation is satisfied

d) this model expand from a Big Bang singularity at $\eta = 0$ where $t = 0$, $a = 0$, $\rho = \infty$

the scale factor starts from 0 at $t = 0$ ($\eta = 0$) and increases with no bounds: it means that there is not a big crunch, the universe expands forever.

This is what we expect for an open universe with negative curvature $K = -1$



PROBLEM 3

the variational principle states that the world line of a free particle extremizes the proper time:

a) $\Delta\tau = \int d\tau = \int d\sigma \frac{d\tau}{d\sigma}$ where $L = \frac{d\tau}{d\sigma}$ is the lagrangian

$$L = \frac{d\tau}{d\sigma} = \frac{ds}{d\sigma} = \left[g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right]^{\frac{1}{2}}$$

$ds^2 = y^{-2}(dx^2+dy^2) = g_{\alpha\beta}\,dx^\alpha dx^\beta$ $(y>0)$ is the metric

$$L = \left[y^{-2}\left(\frac{dx}{d\sigma}\right)^2 + y^{-2}\left(\frac{dy}{d\sigma}\right)^2\right]^{\frac{1}{2}} = \left\{\left[y^{-2}\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right)\right]\right\}^{\frac{1}{2}}$$

b) euler - lagrange equations:

(1) $$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{\partial x}{\partial\sigma}\right)}\right) + \frac{\partial L}{\partial x} = 0$$

$$\left.\begin{aligned}\frac{\partial L}{\partial\left(\frac{\partial x}{\partial\sigma}\right)} &= \frac{1}{2}(L^2)^{-\frac{1}{2}}\,2y^{-2}\frac{dx}{d\sigma}\\ \frac{\partial L}{\partial x} &= 0\end{aligned}\right\} \quad \frac{d}{d\sigma}\left[\frac{1}{Ly^2}\frac{dx}{d\sigma}\right] = 0$$

$$L = \frac{ds}{d\sigma} \rightarrow \frac{d}{d\sigma}\left[\frac{1}{y^2}\frac{d\sigma}{ds}\frac{dx}{d\sigma}\right] = 0 \quad \text{and} \quad \frac{d}{d\sigma}\left[\frac{1}{y^2}\frac{dx}{ds}\right] = 0$$

multiplying by $\frac{d\sigma}{ds}$: $\frac{d}{ds}\left[\frac{1}{y^2}\frac{dx}{ds}\right] = 0$

$$\frac{1}{y^2}\frac{d^2x}{ds^2} - \frac{2}{y^3}\frac{dy}{ds}\frac{dx}{ds} = 0 \rightarrow \boxed{\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0}$$

(2) $$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{\partial y}{\partial\sigma}\right)}\right) + \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial\left(\frac{\partial y}{\partial\sigma}\right)} = \frac{1}{2}(L^2)^{-\frac{1}{2}}\,y^{-2}\,2\frac{dy}{d\sigma} = \frac{1}{L}y^{-2}\frac{dy}{d\sigma}$$

$$\frac{\partial L}{\partial y} = \frac{1}{2}(L^2)^{-\frac{1}{2}}(-2)y^{-3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] = -\frac{1}{L}y^{-3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]$$

$$\frac{d}{d\sigma}\left(\frac{1}{Ly^2}\frac{dy}{d\sigma}\right) + \frac{1}{Ly^3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] = 0 \qquad L = \frac{ds}{d\sigma}$$

multiplying by $\frac{d\sigma}{ds}$:

$$\frac{d}{ds}\left(\frac{1}{y^2}\frac{dy}{ds}\right) + \frac{1}{y^3}\left(\frac{d\sigma}{ds}\right)^2\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] = 0$$

$$\frac{1}{y^2}\frac{d^2y}{ds^2} + \left(\frac{dy}{ds}\right)^2\left(-\frac{2}{y^3}\right) + \frac{1}{y^3}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = 0$$

$$\frac{d^2y}{ds^2} - \frac{2}{y}\left(\frac{dy}{ds}\right)^2 + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 + \frac{1}{y}\left(\frac{dy}{ds}\right)^2 \Big] = 0$$

$$\frac{d^2y}{ds^2} - \frac{2}{y}\left(\frac{dy}{ds}\right)^2 + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 + \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0$$

$$\boxed{\ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0}$$

c) the geodesic equations, from point b) are:

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \quad \text{and} \quad \frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0$$

comparing with the general form of the geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \qquad ds = d\tau$$

(1) $$\frac{d^2x}{ds^2} = \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} \qquad \begin{matrix}\alpha = x\\ \beta = x\\ \gamma = y\end{matrix} \rightarrow \boxed{\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}}$$

(2) $$\frac{d^2y}{ds^2} = -\frac{1}{y}\frac{dx}{ds}\frac{dx}{ds} + \frac{1}{y}\frac{dy}{ds}\frac{dy}{ds}$$

$$\begin{matrix}\alpha = y\\ \beta = x\\ \gamma = x\end{matrix} \rightarrow \boxed{\Gamma^y_{xx} = \frac{1}{y}}$$

$$\begin{matrix}\alpha = y\\ \beta = y\\ \gamma = y\end{matrix} \rightarrow \boxed{\Gamma^y_{yy} = -\frac{1}{y}}$$

d) the metric is invariant under translations $x \to x + c$ and $\xi^\alpha = (1,0)$ is a killing vector

$$\frac{\partial g^{\alpha\beta}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial L}{\partial x} = 0$$

Euler Lagrange equations for x

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)}\right) + \frac{\partial L}{\partial x} = 0$$

$$\rightarrow \frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)}\right) = 0 \quad \text{and} \quad \frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)} = \text{const.}$$

$$\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)} = \frac{1}{2}\left(L^2\right)^{\frac{1}{2}}\left(g_{1\beta}\frac{dx^\beta}{d\sigma}\right) = \frac{1}{2L} g_{1\beta}\frac{dx^\beta}{d\tau}\frac{d\tau}{d\sigma}$$

$$= \frac{1}{2} g_{1\beta}\frac{dx^\beta}{d\tau} = \text{const}$$

$$\rightarrow g_{\alpha\beta}\,\xi^\alpha u^\beta = \text{const}$$

$$g_{\alpha\beta}\,\xi^\alpha u^\beta = g_{xx}\frac{dx}{d\tau} = \frac{1}{y^2}\frac{dx}{d\tau}$$

this is a conserved quantity and expresses the x-component of the linear momentum.

e) the metric is: $ds^2 = \frac{1}{y^2}dx^2 + \frac{1}{y^2}dy^2$

$\vec{u}\cdot\vec{u} = 1$

$\vec{u}\cdot\vec{u} = g_{\alpha\beta}u^\alpha u^\beta = \frac{1}{y^2}(\dot{x}^2+\dot{y}^2) = 1 \rightarrow \dot{x}^2+\dot{y}^2 = y^2$

the first geodesic equation found in point b is:

$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \rightarrow \frac{\ddot{x}}{\dot{x}} = 2\frac{\dot{y}}{y}$

integrating:

$\ln\dot{x} = 2\ln y + A \rightarrow \ln\dot{x} = \ln y^2 + A$

and $\dot{x} = Ay^2$

$\left.\begin{matrix}\dot{x}^2+\dot{y}^2 = y^2\\ \dot{x} = Ay^2\end{matrix}\right\} \rightarrow A^2y^4 + \dot{y}^2 = y^2$

$\dot{y}^2 = y^2(1-A^2y^2)$

$\dot{y} = \pm y(1-A^2y^2)^{\frac{1}{2}}$

$\frac{dx}{dy} = \frac{dx}{ds}\cdot\frac{ds}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{\pm Ay^2}{y\sqrt{1-A^2y^2}} = \pm\frac{Ay}{A\sqrt{\frac{1}{A^2}-y^2}}$

$dx = \pm\frac{y}{\sqrt{\frac{1}{A}-y^2}}dy$

$\int\frac{y\,dy}{\sqrt{r^2-y^2}} = -\sqrt{r^2-y^2}$

here $r^2 = \frac{1}{A^2}$

integrating

$x = x_0 \mp\sqrt{\frac{1}{A^2}-y^2}$

$(x-x_0)^2 = \frac{1}{A^2} - y^2$

and $\boxed{(x-x_0)^2 + y^2 = \frac{1}{A^2}}$

since $y>0$ this equation represents semi-circles in the upper half plane, centered in $(x_0, 0)$ and radius $r = \frac{1}{A}$