Example Exam
WMPH009-05

15:00 - 17:00
Friday November 1, 2024
Exam Hall 4 V3 - X21 Blauwborgje 4

Indicate at the first page clearly your name and student number.
You can obtain a maximum of **(50 + 40 = 90 pt)**

Question 1: Cosmology (50 pt)

Consider the Friedman equation (we take $c = G = 1$)

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = 1\,, \tag{1}$$

where $a = a(t)$ is the scale factor, $\dot{a} = \frac{da}{dt}$ and $\rho = \rho(t)$ is the total energy density.

For a universe with pressureless matter the total energy density $\rho = \rho(t)$ is given by

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3\,, \tag{2}$$

where $\rho_0 = \rho(t_0)\,, a_0 = a(t_0)$ and t_0 is the present-day time

(1.1) Give the definition of ρ_{crit} and show that it is given by the expression (10 pt)

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi} \qquad \text{with} \qquad H_0 = \frac{\dot{a}(t_0)}{a(t_0)}\,. \tag{3}$$

(1.2) Show that the Friedman equation, taken at $t = t_0$, can be used to solve for $a_0 = a(t_0)$ in terms of H_0 and $\Omega \equiv \rho_0/\rho_{\text{crit}}$ as follows: (10 pt)

$$a_0 = (H_0)^{-1}\,(1-\Omega)^{-1/2}\,. \tag{4}$$

(1.3) Use the result (4) to show that the Friedman equation, for general t, can be written as (10 pt)

$$\dot{a}^2 - 1 = \frac{A^2}{a} \qquad \text{with} \qquad A^2 = \frac{\Omega}{H_0(1-\Omega)^{3/2}}\,. \tag{5}$$

(1.4) Show that equation (5) is solved by (10 pt)

$$a(\eta) = \frac{1}{2}A^2(\cosh\eta - 1)\,, \qquad\qquad t(\eta) = \frac{1}{2}A^2(\sinh\eta - \eta) \tag{6}$$

in terms of a parameter η.

(1.5) Explain whether the solution (6) describes a flat, open or closed universe that starts with a Big Bang at $t = 0$. (10 pt)

Question 2: From General Relativity to Special Relativity (40 pt)

Under a general coordinate transformation $x'^\alpha = x'^\alpha(x^\beta)$ the components a^α of a vector field $\mathbf{a}$ and the components b_α of a vector field $\mathbf{b}$ transform as

$$a'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta}a^\beta\,, \qquad b'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}b_\beta\,. \tag{7}$$

(2.1) Show that the metric tensor transforms under general coordinate transformations as

$$g'_{\alpha\beta}(x) = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\,g_{\gamma\delta}(x) \tag{8}$$

by requiring that this tensor transforms in the same way as the product of two vectors with a lower index. (10 pt)

(2.2) In general relativity the scalar product between two vector fields $\mathbf{v}$(x) and $\mathbf{w}$(x) is defined by

$$\mathbf{v}(x)\cdot\mathbf{w}(x) = g_{\alpha\beta}(x)v^\alpha(x)w^\beta(x)\,, \tag{9}$$

where $g_{\alpha\beta}$ is the spacetime metric. Show that the scalar product (9) is invariant under the general coordinate transformations given in (7) and (8). (10 pt)

(2.3) Show that under a general coordinate transformation $x'^\alpha = (\Lambda^{-1})^\alpha{}_\beta x^\beta$, with Λ^{-1} a constant invertable matrix (with inverse given by Λ), the components b_α of a vector field $\mathbf{b}$ transform as (10 pt)

$$b'_\alpha = \Lambda^\beta{}_\alpha b_\beta\,. \tag{10}$$

(2.4) We consider a local inertial frame at a point P where the metric $g_{\alpha\beta}$ at P is the Minkowski spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta}$ and where the first derivative of the metric $g_{\alpha\beta}$ at P vanishes. Using eq. (8), show that the condition for which the Minkowski spacetime metric is invariant under the general coordinate transformations defined by the matrix $\Lambda^{\alpha}{}_{\beta}$ given in (2.3), i.e. the condition for which $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$, is given by the equations

$$\left(\Lambda^T\right)^{\alpha}{}_{\gamma}\left(\Lambda\right)^{\gamma}{}_{\beta} = \mathbb{1}^{\alpha}{}_{\beta} \qquad \text{with} \qquad (\Lambda^T)^{\alpha}{}_{\beta} \equiv \eta^{\alpha\gamma}\eta_{\beta\delta}\Lambda^{\delta}{}_{\gamma}\,. \tag{11}$$

How are these special general coordinate transformations called? (10 pt)